Example 29. (review) Find the general solution of $y''' - 3y' + 2y = 0$.

Solution. The characteristic polynomial $p(D) = D^3 - 3D + 2 = (D-1)^2(D+2)$ has roots $1, 1, -2$.
By Theorem 24, the general solution is $y(x) = (C_1 + C_2 x)e^x + C_2 e^{-2x}$.

Inhomogeneous linear DEs with constant coefficients

Example 30. ("warmup") Find the general solution of $y'' + 4y = 12x$.

Solution. Here, $p(D) = D^2 + 4$, which has roots $\pm 2i$.

Hence, the general solution is $y(x) = y_p(x) + C_1\cos(2x) + C_2\sin(2x)$. It remains to find a particular solution y_p.

Noting that $D^2 \cdot (12x) = 0$, we apply D^2 to both sides of the DE.

We get $D^2(D^2+4) \cdot y = 0$, which is a homogeneous linear DE! Its general solution is $C_1 + C_2 x + C_3\cos(2x) + C_4\sin(2x)$. In particular, y_p is of this form for some choice of $C_1, \ldots, C_4$.

It simplifies our life to note that there has to be a particular solution of the simpler form $y_p = C_1 + C_2 x$.

[Why?! Because we know that $C_3\cos(2x) + C_4\sin(2x)$ can be added to any particular solution.]

It only remains to find appropriate values C_1, C_2 such that $y_p'' + 4y_p = 12x$. Since $y_p'' + 4y_p = 4C_1 + 4C_2 x$, comparing coefficients yields $4C_1 = 0$ and $4C_2 = 12$, so that $C_1 = 0$ and $C_2 = 3$. In other words, $y_p = 3x$.

Therefore, the general solution to the original DE is $y(x) = 3x + C_1\cos(2x) + C_2\sin(2x)$.

Example 31. ("warmup") Find the general solution of $y'' + 4y' + 4y = e^{3x}$.

Solution. This is $p(D)y = e^{3x}$ with $p(D) = D^2 + 4D + 4 = (D+2)^2$.

Hence, the general solution is $y(x) = y_p(x) + (C_1 + C_2 x)e^{-2x}$. It remains to find a particular solution y_p.

Note that $(D-3)e^{3x} = 0$. Hence, we apply $(D-3)$ to the DE to get $(D-3)(D+2)^2 y = 0$.

This homogeneous linear DE has general solution $(C_1 + C_2 x)e^{-2x} + C_3 e^{3x}$. We conclude that the original DE must have a particular solution of the form $y_p = C_3 e^{3x}$.

To determine the value of C_3, we plug into the original DE: $y_p'' + 4y_p' + 4y_p = (9 + 4 \cdot 3 + 4)C_3 e^{3x} \overset{!}{=} e^{3x}$. Hence, $C_3 = 1/25$. In conclusion, the general solution is $y(x) = (C_1 + C_2 x)e^{-2x} + \frac{1}{25}e^{3x}$.

Comment. See Example 33 for the same solution in more compact form.

We found a recipe for solving nonhomogeneous linear DEs with constant coefficients.

Our approach works for $p(D)y = f(x)$ whenever the right-hand side $f(x)$ is the solution of some homogeneous linear DE with constant coefficients: $q(D)f(x) = 0$

Theorem 32. (method of undetermined coefficients) To find a particular solution y_p to an inhomogeneous linear DE with constant coefficients $p(D)y = f(x)$:

- Find $q(D)$ so that $q(D)f(x) = 0$. [This does not work for all $f(x)$.]
- Let $r_1, \ldots, r_n$ be the ("old") roots of the polynomial $p(D)$.
  Let $s_1, \ldots, s_m$ be the ("new") roots of the polynomial $q(D)$.
- It follows that y_p solves the **homogeneous** DE $q(D)\,p(D)y = 0$.
  The characteristic polynomial of this DE has roots $r_1, \ldots, r_n, s_1, \ldots, s_m$.
  Let $v_1, \ldots, v_m$ be the "new" solutions (i.e. not solutions of the "old" $p(D)y = 0$).
  By plugging into $p(D)y_p = f(x)$, we find (unique) C_i so that $y_p = C_1 v_1 + \ldots + C_m v_m$.

Because of the final step, this approach is often called **method of undetermined coefficients**.

For which $f(x)$ does this work? By Theorem 24, we know exactly which $f(x)$ are solutions to homogeneous linear DEs with constant coefficients: these are linear combinations of exponentials $x^j e^{rx}$ (which includes $x^j e^{ax}\cos(bx)$ and $x^j e^{ax}\sin(bx)$).

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Example 33. (again) Determine the general solution of $y'' + 4y' + 4y = e^{3x}$.

Solution. The "old" roots are $-2, -2$. The "new" roots are 3. Hence, there has to be a particular solution of the form $y_p = Ce^{3x}$. To find the value of C, we plug into the DE.

$y_p'' + 4y_p' + 4y_p = (9 + 4 \cdot 3 + 4)Ce^{3x} \overset{!}{=} e^{3x}$. Hence, $C = 1/25$.

Therefore, the general solution is $y(x) = (C_1 + C_2 x)e^{-2x} + \frac{1}{25}e^{3x}$.

Example 34. Determine the general solution of $y'' + 4y' + 4y = 7e^{-2x}$.

Solution. The "old" roots are $-2, -2$. The "new" roots are -2. Hence, there has to be a particular solution of the form $y_p = Cx^2 e^{-2x}$. To find the value of C, we plug into the DE.

$y_p' = C(-2x^2 + 2x)e^{-2x}$

$y_p'' = C(4x^2 - 8x + 2)e^{-2x}$

$y_p'' + 4y_p' + 4y_p = 2Ce^{-2x} \overset{!}{=} 7e^{-2x}$

It follows that $C = 7/2$, so that $y_p = \frac{7}{2}x^2 e^{-2x}$. The general solution is $y(x) = \left(C_1 + C_2 x + \frac{7}{2}x^2\right)e^{-2x}$.

Example 35. Determine a particular solution of $y'' + 4y' + 4y = 2e^{3x} - 5e^{-2x}$.

Solution. Write the DE as $Ly = 2e^{3x} - 5e^{-2x}$ where $L = D^2 + 4D + 4$. Instead of starting all over, recall that in Example 33 we found that $y_1 = \frac{1}{25}e^{3x}$ satisfies $Ly_1 = e^{3x}$. Also, in Example 34 we found that $y_2 = \frac{7}{2}x^2 e^{-2x}$ satisfies $Ly_2 = 7e^{-2x}$.

By linearity, it follows that $L(Ay_1 + By_2) = ALy_1 + BLy_2 = Ae^{3x} + 7Be^{-2x}$.

To get a particular solution y_p of our DE, we need $A = 2$ and $7B = -5$.

Hence, $y_p = 2y_1 - \frac{5}{7}y_2 = \frac{2}{25}e^{3x} - \frac{5}{2}x^2 e^{-2x}$.

Example 36. (homework) Determine the general solution of $y'' - 2y' + y = 5\sin(3x)$.

Solution. Since $D^2 - 2D + 1 = (D-1)^2$, the "old" roots are $1, 1$. The "new" roots are $\pm 3i$. Hence, there has to be a particular solution of the form $y_p = A\cos(3x) + B\sin(3x)$.

To find the values of A and B, we plug into the DE.

$y_p' = -3A\sin(3x) + 3B\cos(3x)$

$y_p'' = -9A\cos(3x) - 9B\sin(3x)$

$y_p'' - 2y_p' + y_p = (-8A - 6B)\cos(3x) + (6A - 8B)\sin(3x) \overset{!}{=} 5\sin(3x)$

Equating the coefficients of $\cos(x)$, $\sin(x)$, we obtain the two equations $-8A - 6B = 0$ and $6A - 8B = 5$.

Solving these, we find $A = \frac{3}{10}$, $B = -\frac{2}{5}$. Accordingly, a particular solution is $y_p = \frac{3}{10}\cos(3x) - \frac{2}{5}\sin(3x)$.

The general solution is $y(x) = \frac{3}{10}\cos(3x) - \frac{2}{5}\sin(3x) + (C_1 + C_2 x)e^x$.

Example 37. (homework) What is the shape of a particular solution of $y'' + 4y' + 4y = x\cos(x)$?

Solution. The "old" roots are $-2, -2$. The "new" roots are $\pm i, \pm i$. Hence, there has to be a particular solution of the form $y_p = (C_1 + C_2 x)\cos(x) + (C_3 + C_4 x)\sin(x)$.

Continuing to find a particular solution. To find the value of the C_j's, we plug into the DE.

$y_p' = (C_2 + C_3 + C_4 x)\cos(x) + (C_4 - C_1 - C_2 x)\sin(x)$

$y_p'' = (2C_4 - C_1 - C_2 x)\cos(x) + (-2C_2 - C_3 - C_4 x)\sin(x)$

$$y_p'' + 4y_p' + 4y_p = (3C_1 + 4C_2 + 4C_3 + 2C_4 + (3C_2 + 4C_4)x)\cos(x) + (-4C_1 - 2C_2 + 3C_3 + 4C_4 + (-4C_2 + 3C_4)x)\sin(x) \overset{!}{=} x\cos(x).$$

Equating the coefficients of $\cos(x)$, $x\cos(x)$, $\sin(x)$, $x\sin(x)$, we get the equations $3C_1 + 4C_2 + 4C_3 + 2C_4 = 0$, $3C_2 + 4C_4 = 1$, $-4C_1 - 2C_2 + 3C_3 + 4C_4 = 0$, $-4C_2 + 3C_4 = 0$.

Solving (this is tedious!), we find $C_1 = -\frac{4}{125}$, $C_2 = \frac{3}{25}$, $C_3 = -\frac{22}{125}$, $C_4 = \frac{4}{25}$.

Hence, $y_p = \left(-\frac{4}{125} + \frac{3}{25}x\right)\cos(x) + \left(-\frac{22}{125} + \frac{4}{25}x\right)\sin(x)$.

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