Review: Linear DEs

A linear DE of order n is of the form $y^{(n)} + p_{n-1}(x)\, y^{(n-1)} + \ldots + p_1(x) y' + p_0(x) y = f(x)$.

- In terms of $D = \frac{\mathrm{d}}{\mathrm{d}x}$, the DE becomes: $Ly = f(x)$ with $L = D^n + p_{n-1}(x) D^{n-1} + \ldots + p_1(x) D + p_0(x)$.
  Comment. L is called a (linear) differential operator.
- The inclusion of the $f(x)$ term makes $Ly = f(x)$ an **inhomogeneous** linear DE.
- $Ly = 0$ is the corresponding **homogeneous** DE.
  - If y_1 and y_2 are solutions to the homogeneous DE, then so is any linear combination $C_1 y_1 + C_2 y_2$.
  - **(general solution of the homogeneous DE)** There are n solutions $y_1, y_2, \ldots, y_n$, such that every solution is of the form $C_1 y_1 + \ldots + C_n y_n$. [These n solutions necessarily are **independent**.]
- To find the general solution of the inhomogeneous DE, we only need to find a single solution y_p (called a **particular solution**). Then the general solution is $y_p + y_h$, where y_h is the general solution of the homogeneous DE.

Example 20. Consider the following DEs. If linear, write them in operator form as $Ly = f(x)$.

(a) $y'' = xy$

(b) $x^2 y'' + xy' = (x^2 + 4) y + x(x^2 + 3)$

(c) $y'' = y' + 2y + 2(1 - x - x^2)$

(d) $y'' = y' + 2y + 2(1 - x - y^2)$

Solution.

(a) This is a homogeneous linear DE: $\underbrace{(D^2 - x)}_{L} y = \underbrace{0}_{f(x)}$

Note. This is known as the Airy equation, which we will meet again later. The general solution is of the form $C_1 y_1(x) + C_2 y_2(x)$ for two special solutions y_1, y_2. [In the literature, one usually chooses functions called $\mathrm{Ai}(x)$ and $\mathrm{Bi}(x)$ as y_1 and y_2. See: https://en.wikipedia.org/wiki/Airy_function]

(b) This is an inhomogeneous linear DE: $\underbrace{(x^2 D^2 + xD - (x^2 + 4))}_{L} y = \underbrace{x(x^2 + 3)}_{f(x)}$

Note. The corresponding homogeneous DE is an instance of the "modified Bessel equation" $x^2 y'' + xy' - (x^2 + \alpha^2) y = 0$, namely the case $\alpha = 2$. Because they are important for applications (but cannot be written in terms of familiar functions), people have introduced names for two special solutions of this differential equation: $I_\alpha(x)$ and $K_\alpha(x)$ (called modified Bessel functions of the first and second kind).

It follows that the general solution of the modified Bessel equation is $C_1 I_\alpha(x) + C_2 K_\alpha(x)$.

In our case. The general solution of the homogeneous DE (which is the modified Bessel equation with $\alpha = 2$) is $C_1 I_2(x) + C_2 K_2(x)$. On the other hand, we can (do it!) easily check (this is coming from nowhere at this point!) that $y_p = -x$ is a particular solution to the original inhomogeneous DE.

It follows that the general solution to the original DE is $C_1 I_2(x) + C_2 K_2(x) - x$.

(c) This is an inhomogeneous linear DE: $\underbrace{(D^2 - D - 2)}_{L} y = \underbrace{2(1 - x - x^2)}_{f(x)}$

Note. We will recall in Example 21 that the corresponding homogeneous DE $(D^2 - D - 2) y = 0$ has general solution $C_1 e^{2x} + C_2 e^{-x}$. On the other hand, we can check that $y_p = x^2$ is a particular solution of the original inhomogeneous DE. (Do you recall from DE1 how to find this particular solution?)

It follows that the general solution to the original DE is $x^2 + C_1 e^{2x} + C_2 e^{-x}$.

(d) This is not a linear DE because of the term y^2. It cannot be written in the form $Ly = f(x)$.