Example 11. (review) $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 14 \end{bmatrix}$ whereas $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$.

Review: Examples of differential equations we can solve

Let's start with one of the simplest (and most fundamental) differential equations (DE). It is **first-order** (only a first derivative) and **linear** (with constant coefficients).

Example 12. Solve $y' = 3y$.

Solution. $y(x) = Ce^{3x}$

Check. Indeed, if $y(x) = Ce^{3x}$, then $y'(x) = 3Ce^{3x} = 3y(x)$.

Comment. Recall we can always easily check whether a function solves a differential equation. This means that (although you might be unfamiliar with the techniques for solving) you can use computer algebra systems like Sage to solve differential equations without trust issues.

To describe a unique solution, additional constraints need to be imposed.

Example 13. Solve the **initial value problem** (IVP) $y' = 3y$, $y(0) = 5$.

Solution. This has the unique solution $y(x) = 5e^{3x}$.

The following is a **non-linear** differential equation. In general, such equations are much more complicated than linear ones. We can solve this particular one because it is **separable**.

Example 14. Solve $y' = xy^2$.

Solution. This DE is separable: $\frac{1}{y^2}\mathrm{d}y = x\,\mathrm{d}x$. Integrating, we find $-\frac{1}{y} = \frac{1}{2}x^2 + C$.

Hence, $y = -\frac{1}{\frac{1}{2}x^2 + C} = \frac{2}{D - x^2}$.

[Here, $D = -2C$ but that relationship doesn't matter; it only matters that the solution has a free parameter.]

Note. Note that we did not find the solution $y = 0$ (lost when dividing by y^2). It is called a **singular solution** because it is not part of the **general solution** (the one-parameter family found above). [Although, we can obtain it from the general solution by letting $D \to \infty$.]

Check. Compute y' and verify that the DE is indeed satisfied.

Review: Linear first-order DEs

The most general first-order linear DE is $P(x)y' + Q(x)y + R(x) = 0$.

By dividing by $P(x)$ and rearranging, we can always write it in the form $y' = a(x)y + f(x)$.

We will recall next time that we can always solve this DE.

The corresponding **homogeneous** linear DE is $y' = a(x)y$.

Important comment. Write $D = \frac{\mathrm{d}}{\mathrm{d}x}$. Then we can write $y' - a(x)y = f(x)$ as $Ly = f(x)$ where $L = D - a(x)$.
The corresponding homogeneous DE is simply $Ly = 0$.

Armin Straub
straub@southalabama.edu

Solving linear first-order DEs using variation of constants

The following DE is linear and first-order (but not with constant coefficients).

Example 15. (homework) Solve $y' = x^2 y$.

Solution. This DE is separable as well: $\frac{1}{y}\mathrm{d}y = x^2\,\mathrm{d}x$ (note that we just lost the solution $y = 0$).

Integrating gives $\ln|y| = \frac{1}{3}x^3 + A$, so that $|y| = e^{\frac{1}{3}x^2 + A}$. Since the RHS is never zero, we must have either $y = e^{\frac{1}{3}x^2 + A}$ or $y = -e^{\frac{1}{3}x^2 + A}$.

Hence $y = \pm e^A e^{\frac{1}{3}x^3} = Ce^{\frac{1}{3}x^3}$ (with $C = \pm e^A$). Note that $C = 0$ corresponds to the singular solution $y = 0$.

In summary, the general solution is $y = Ce^{\frac{1}{3}x^3}$ (with C any real number).

Check. Compute y' and verify that the DE is indeed satisfied.

As in the previous example, we can immediately solve any **homogeneous** linear first-order DE:

Example 16. Solve $y' = a(x)y$.

Solution. Proceeding as in the previous example, we find $y(x) = Ce^{\int a(x)\mathrm{d}x}$.

Check. Compute y' and verify that the DE is indeed satisfied.