Final Exam – Practice

MATH 332 — Differential Equations II
Final Exam: Wednesday, December 13, 2023

Please print your name:

Bonus challenge. Let me know about any typos you spot in the posted solutions (or lecture sketches). Any typo, that is not yet fixed by the time you send it to me, is worth a bonus point.

Problem 1. The final exam will be comprehensive, that is, it will cover the material of the whole semester.

(a) Do the practice problems for both midterms.

(b) Retake both midterm exams.

(c) Do the problems below. (Solutions are posted.)

Problem 2. For $t \geqslant 0$ and $x \in [0, 4]$, consider the heat flow problem:
$$\begin{aligned} u_t &= 2u_{xx} + e^{-x/2} \\ u_x(0,t) &= 3 \\ u(4,t) &= -2 \\ u(x,0) &= f(x) \end{aligned}$$

Determine the steady-state solution and spell out equations characterizing the transient solution.

Solution. We look for a solution of the form $u(x,t) = v(x) + w(x,t)$, where $v(x)$ is the steady-state solution and where the transient solution $w(x,t)$ tends to zero as $t \to \infty$ (as do its derivatives).

- Plugging into (PDE), we get $w_t = 2v'' + 2w_{xx} + e^{-x/2}$. Letting $t \to \infty$, this becomes $0 = 2v'' + e^{-x/2}$.
- Plugging into (BC), we get $w_x(0,t) + v'(0) = 3$ and $w(4,t) + v(4) = -2$.
  Letting $t \to \infty$, these become $v'(0) = 3$ and $v(4) = -2$.
- Solving $0 = 2v'' + e^{-x/2}$, we find
  $$v(x) = \iint -\frac{1}{2} e^{-x/2} \mathrm{d}x \mathrm{d}x = \int e^{-x/2} \mathrm{d}x + C = -2e^{-x/2} + Cx + D.$$
  The boundary conditions $v'(0) = 3$ and $v(4) = -2$ imply $C = 2$ and $-2e^{-2} + 8 + D = -2$.
  and therefore the steady-state solution $v(x) = -2e^{-x/2} + 2x - 10 + 2e^{-2}$.

On the other hand, the transient solution $w(x,t)$ is characterized as the unique solution to:

$$\begin{aligned} & w_t = 2w_{xx} \\ & w_x(0,t) = 0, \quad w(4,t) = 0 \\ & w(x,0) = f(x) - v(x) \end{aligned}$$

Note. We know how to solve this homogeneous heat equation (see Problem 8) using separation of variables.

Problem 3. Using a step size of $h = \frac{1}{2}$, discretize the Dirichlet problem
$$\begin{aligned} u_{xx} + u_{yy} &= 0 \\ u(x,0) &= 3 \\ u(x,2) &= 5 \\ u(0,y) &= 1 \\ u(1,y) &= 2 \end{aligned}$$
where $x \in (0,1)$ and $y \in (0,2)$.

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Spell out a system of linear equations for the resulting lattice points. Do not solve that system.

(Note that, for the Dirichlet problem as well as for our discretization, it doesn't matter that the boundary conditions aren't well-defined at the corners.)

Solution. Note that our rectangle has side lengths 1 (in x direction) and 2 (in y direction). With a step size of $h=\frac{1}{2}$ we therefore get $3\cdot 5$ lattice points, namely the points

$$u_{m,n}=u(mh,nh), \quad m\in\{0,1,2\},\ n\in\{0,1,2,3,4\}.$$

Further note that the boundary conditions determine the values of $u_{m,n}$ if $m=0$ or $m=2$ as well as if $n=0$ or $n=4$. This leaves $1\cdot 3=3$ points at which we need to determine the value of $u_{m,n}$.

If we approximate $u_{xx}+u_{yy}$ by $\frac{1}{h^2}[u(x+h,y)+u(x-h,y)+u(x,y+h)+u(x,y-h)-4u(x,y)]$ then, in terms of our lattice points, the equation $u_{xx}+u_{yy}=0$ translates into

$$u_{m+1,n}+u_{m-1,n}+u_{m,n+1}+u_{m,n-1}-4u_{m,n}=0.$$

Spelling out these equation for each $m=1$ and $n\in\{1,2,3\}$, we get 3 equations for our 3 unknown values:

$$\begin{aligned}
\underbrace{u_{2,1}}_{=2}+\underbrace{u_{0,1}}_{=1}+u_{1,2}+\underbrace{u_{1,0}}_{=3}-4u_{1,1} &= 0\\
\underbrace{u_{2,2}}_{=2}+\underbrace{u_{0,2}}_{=1}+u_{1,3}+u_{1,1}-4u_{1,2} &= 0\\
\underbrace{u_{2,3}}_{=2}+\underbrace{u_{0,3}}_{=1}+\underbrace{u_{1,4}}_{=5}+u_{1,2}-4u_{1,3} &= 0
\end{aligned}$$

In matrix-vector form, these linear equations take the form:

$$\begin{bmatrix} -4 & 1 & 0\\ 1 & -4 & 1\\ 0 & 1 & -4 \end{bmatrix}\begin{bmatrix} u_{1,1}\\ u_{1,2}\\ u_{1,3} \end{bmatrix}=\begin{bmatrix} -6\\ -3\\ -8 \end{bmatrix}$$

Problem 4. Find all eigenfunctions and eigenvalues of $y''+\lambda y=0$, $y(0)=0$, $y(2)=0$ in the case $\lambda>0$.

Solution. Write $\lambda=\rho^2$. Then $y(x)=A\cos(\rho x)+B\sin(\rho x)$. $y(0)=A\overset{!}{=}0$. Using this, $y(2)=B\sin(2\rho)\overset{!}{=}0$. Since $B\neq 0$ for eigenfunctions, we need $\sin(2\rho)=0$. This happens if $2\rho=n\pi$ for an integer n.

Consequently, we find the eigenvalues $\lambda=\left(\frac{n\pi}{2}\right)^2$, where $n=1,2,3,\ldots$ (we exclude $n=0$ because $\lambda>0$), with corresponding eigenfunctions $y(x)=\sin\frac{n\pi x}{2}$.

Problem 5. Find all eigenfunctions and eigenvalues of

$$y''+\lambda y=0, \quad y'(0)=0, \quad y(3)=0.$$

Solution. We distinguish three cases:

- $\boldsymbol{\lambda<0}$. The characteristic roots are $\pm r=\pm\sqrt{-\lambda}$ and the general solution to the DE is $y(x)=Ae^{rx}+Be^{-rx}$. Then $y'(0)=Ar-Br=0$ implies $B=A$, so that $y(3)=A(e^{3r}+e^{-3r})$. Since $e^{3r}+e^{-3r}>0$, we see that $y(3)=0$ only if $A=0$. So there is no solution for $\lambda<0$.

- $\boldsymbol{\lambda=0}$. The general solution to the DE is $y(x)=A+Bx$. Then $y'(0)=0$ implies $B=0$, and it follows from $y(3)=A=0$ that $\lambda=0$ is not an eigenvalue.

$\lambda > 0$. The characteristic roots are $\pm i\sqrt{\lambda}$. So, with $r = \sqrt{\lambda}$, the general solution is $y(x) = A\cos(rx) + B\sin(rx)$. $y'(0) = Br = 0$ implies $B = 0$. Then $y(3) = A\cos(3r) = 0$. Note that $\cos(3r) = 0$ is true if and only if $3r = \frac{\pi}{2} + n\pi = \frac{(2n+1)\pi}{2}$ for some integer n. Since $r > 0$, we have $n \geqslant 0$. Correspondingly, $\lambda = r^2 = \left(\frac{(2n+1)\pi}{6}\right)^2$ and $y(x) = \cos\left(\frac{(2n+1)\pi}{6}x\right)$.

In summary, we have that the eigenvalues are $\lambda = \left(\frac{(2n+1)\pi}{6}\right)^2$, with $n = 0, 1, 2, \ldots$ with corresponding eigenfunctions $y(x) = \cos\left(\frac{(2n+1)\pi}{6}x\right)$.

Problem 6. Find all eigenfunctions and eigenvalues of

$$y'' + \lambda y = 0, \quad y'(0) = 0, \quad y'(3) = 0.$$

Solution. To solve this eigenvalue problem, we distinguish three cases:

$\lambda < 0$. Then, the roots are the real numbers $\pm r = \pm\sqrt{-\lambda}$ and the general solution to the DE is $y(x) = Ae^{rx} + Be^{-rx}$. Then $y'(0) = Ar - Br = 0$ implies $B = A$, so that $y'(3) = A(3e^{3r} - 3e^{-3r})$. Since $3e^{3r} - 3e^{-3r} = 0$ only if $r = 0$, we see that $y'(3) = 0$ only if $A = 0$. So there is no solution for $\lambda < 0$.

$\lambda = 0$. Now, the general solution to the DE is $y(x) = A + Bx$. Then $y'(x) = B$ and we see that $y'(0) = 0$ and $y'(3) = 0$ if and only if $B = 0$. It follows that $\lambda = 0$ is an eigenvalue with corresponding eigenfunction $y(x) = 1$ (or any other constant multiple).

$\lambda > 0$. Now, the roots are $\pm i\sqrt{\lambda}$ and $y(x) = A\cos(\sqrt{\lambda}\,x) + B\sin(\sqrt{\lambda}\,x)$. Hence, $y'(x) = -A\sqrt{\lambda}\sin(\sqrt{\lambda}\,x) + B\sqrt{\lambda}\cos(\sqrt{\lambda}\,x)$. $y'(0) = B\sqrt{\lambda} = 0$ implies $B = 0$. Then, $y'(3) = -A\sqrt{\lambda}\sin(3\sqrt{\lambda}) = 0$ if and only if $\sin(3\sqrt{\lambda}) = 0$. The latter is true if and only if $3\sqrt{\lambda} = n\pi$ for some integer n. In that case, $\lambda = \left(\frac{n\pi}{3}\right)^2$ and $y(x) = \cos\left(\frac{n\pi}{3}x\right)$.

In summary, this means that the eigenvalues are $\lambda = \left(\frac{n\pi}{3}\right)^2$, with $n = 0, 1, 2, \ldots$ (why did we include $n = 0$ but excluded $n = -1, -2, \ldots$?!) with corresponding eigenfunctions $y(x) = \cos\left(\frac{n\pi}{3}x\right)$.

Note. Note that the case $n = 0$ corresponds to the eigenvalue $\lambda = 0$ (with eigenfunction $y(x) = 1$).

Comment. There was nothing special about 3. Likewise, we find that

$$y'' + \lambda y = 0, \quad y'(0) = 0, \quad y'(L) = 0$$

has eigenvalues eigenvalues are $\lambda = \left(\frac{n\pi}{L}\right)^2$, with $n = 0, 1, 2, \ldots$ with corresponding eigenfunctions $y(x) = \cos\left(\frac{n\pi}{L}x\right)$.

Problem 7. Find the solution $u(x,t)$, for $0 < x < 3$ and $t \geqslant 0$, to the heat conduction problem

$$u_t = 5u_{xx}, \quad u_x(0,t) = u_x(3,t) = 0, \quad u(x,0) = 7 + 4\cos(\pi x).$$

Derive your solution using separation of variables (at some step you may refer to one of the EVP's above).

Solution.

- Using separation of variables, we look for solutions $u(x,t) = X(x)T(t)$. Plugging into the PDE, we get $X(x)T'(t) = 5X''(x)T(t)$, and so $\frac{X''(x)}{X(x)} = \frac{T'(t)}{5T(t)} = \text{const}$. We thus have $X'' - \text{const}\,X = 0$ and $T' - 5\text{const}\,T = 0$.

- $u_x(0,t) = X'(0)T(t) = 0$ implies $X'(0) = 0$. Likewise, $u_x(3,t) = X'(3)T(t) = 0$ implies $X'(3) = 0$.

- So X solves $X'' + \lambda X = 0$ (we choose $\lambda = -\text{const}$), $X'(0) = 0$, $X'(3) = 0$. We solved this EVP above and found that, up to multiples, the only nonzero solutions of this eigenvalue problem are $X(x) = \cos\left(\frac{n\pi}{3}x\right)$ corresponding to $\lambda = \left(\frac{n\pi}{3}\right)^2$, $n = 0, 1, 2, 3\ldots$.

- T solves $T' + 5\lambda T = 0$, and hence, up to multiples, $T(t) = e^{-5\lambda t} = e^{-\frac{5}{9}\pi^2 n^2 t}$.

- Taken together, we have the solutions $u_n(x,t) = e^{-\frac{5}{9}\pi^2 n^2 t}\cos\left(\frac{n\pi}{3}x\right)$ solving $u_t = 5u_{xx}$ and $u_x(0,t) = u_x(3,t) = 0$.

Note that $u_n(x,0) = \cos\left(\frac{n\pi}{3}x\right)$. In particular, our heat conduction problem is solved by

$$u(x,t) = 7u_0(x,t) + 4u_3(x,t) = 7 + 4e^{-5\pi^2 t}\cos(\pi x).$$

Problem 8. Find the solution $u(x,t)$, for $0 < x < 3$ and $t \geqslant 0$, to the heat conduction problem

$$2u_t = u_{xx}, \quad u_x(0,t) = 0,\ u(3,t) = 0, \quad u(x,0) = 2\cos\left(\frac{\pi x}{2}\right) + 7\cos\left(\frac{3\pi x}{2}\right).$$

Derive your solution using separation of variables (at some step you may refer to one of the EVP's above).

Solution.

- Using separation of variables, we look for solutions $u(x,t) = X(x)T(t)$. Plugging into the PDE, we get $2X(x)T'(t) = X''(x)T(t)$, and so $\frac{X''(x)}{X(x)} = \frac{2T'(t)}{T(t)} = \text{const}$. We thus have $X'' - \text{const}\, X = 0$ and $2T' - \text{const}\, T = 0$.

- $u_x(0,t) = X'(0)T(t) = 0$ implies $X'(0) = 0$. Likewise, $u(3,t) = X(3)T(t) = 0$ implies $X(3) = 0$.

- So X solves $X'' + \lambda X = 0$ (we choose $\lambda = -\text{const}$), $X'(0) = 0$, $X(3) = 0$. We solved this EVP above and found that, up to multiples, the only nonzero solutions of this eigenvalue problem are $X(x) = \cos\left(\frac{(2n+1)\pi}{6}x\right)$ corresponding to $\lambda = \left(\frac{(2n+1)\pi}{6}\right)^2$, $n = 0, 1, 2, 3 \ldots$.

- T solves $2T' + \lambda T = 0$, and hence, up to multiples, $T(t) = e^{-\frac{1}{2}\lambda t} = e^{-\frac{1}{2}\left(\frac{(2n+1)\pi}{6}\right)^2 t}$.

- Taken together, we have the solutions $u_n(x,t) = e^{-\frac{1}{2}\left(\frac{(2n+1)\pi}{6}\right)^2 t}\cos\left(\frac{(2n+1)\pi}{6}x\right)$ solving $2u_t = u_{xx}$ and $u_x(0,t) = u(3,t) = 0$.

Note that $u_n(x,0) = \cos\left(\frac{(2n+1)\pi}{6}x\right)$. In particular, our heat conduction problem is solved by

$$u(x,t) = 2u_1(x,t) + 7u_4(x,t) = 2e^{-\frac{1}{8}\pi^2 t}\cos\left(\frac{\pi x}{2}\right) + 7e^{-\frac{9}{8}\pi^2 t}\cos\left(\frac{3\pi x}{2}\right).$$

Comment. It is not obvious that *every* initial temperature distribution $f(x)$ can be written as an (infinite) superposition of the $u_n(x,0)$. However, such "eigenfunction expansions" are always possible (thus extending what we know about ordinary Fourier series).

Problem 9. Find the solution $u(x,y)$, for $0 < x < 3$ and $0 < y < 2$, to

$$u_{xx} + u_{yy} = 0, \quad u(x,0) = 0, \quad u(x,2) = 0, \quad u(0,y) = \sin(\pi y) + 2\sin(7\pi y), \quad u(3,y) = 0.$$

Derive your solution using separation of variables.

Solution.

- We look for solutions $u(x,y) = X(x)Y(y)$.

  Plugging into (PDE), we get $X''(x)Y(y) + X(x)Y''(y)$, and so $\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = \text{const}$.

We thus have $X'' - \text{const}\, X = 0$ and $Y'' + \text{const}\, Y = 0$.

- The three homogeneous (BC) translate into $Y(0) = 0$, $Y(2) = 0$, $X(3) = 0$.

- So Y solves $Y'' + \lambda Y = 0$ (we choose $\lambda = \text{const}$), $Y(0) = 0$, $Y(2) = 0$.

  We solved this EVP above and found that, up to multiples, the only nonzero solutions of this eigenvalue problem are $Y(y) = \sin\left(\frac{\pi n}{2} y\right)$ corresponding to $\lambda = \left(\frac{\pi n}{2}\right)^2$, $n = 1, 2, 3, \ldots$

- On the other hand, X solves $X'' - \lambda X = 0$, and hence $X(x) = A e^{\sqrt{\lambda} x} + B e^{-\sqrt{\lambda} x} = A e^{\frac{\pi n}{2} x} + B e^{-\frac{\pi n}{2} x}$.

  The condition $X(3) = 0$ implies that $A e^{3\pi n/2} + B e^{-3\pi n/2} = 0$ so that $B = -A e^{3\pi n}$.

  Hence, $X(x) = A\left(e^{\frac{\pi n}{3} x} - e^{3\pi n}\, e^{-\frac{\pi n}{3} x}\right)$.

- Taken together, we have the solutions $u_n(x, y) = \left(e^{\frac{\pi n}{3} x} - e^{3\pi n} e^{-\frac{\pi n}{3} x}\right) \sin\left(\frac{\pi n}{2} y\right)$ solving (PDE)+(BC), with the exception of $u(0, y) = \sin(\pi y) + 2\sin(7\pi y)$.

- At $x = 0$, $u_n(0, y) = (1 - e^{3\pi n}) \sin\left(\frac{\pi n}{2} y\right)$.

  Consequently, taking the proper combination of $u_2(x, y)$ and $u_{14}(x, y)$, (PDE)+(BC) is solved by

$$\begin{aligned} u(x, y) &= \frac{1}{1 - e^{6\pi}} u_2(x, y) + \frac{2}{1 - e^{42\pi}} u_{14}(x, y) \\ &= \frac{e^{2\pi x/3} - e^{6\pi} e^{-2\pi x/3}}{1 - e^{6\pi}} \sin(\pi y) + 2 \frac{e^{14\pi x/3} - e^{42\pi} e^{-14\pi x/3}}{1 - e^{42\pi}} \sin(7\pi y). \end{aligned}$$

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