Inhomogeneous LDEs with constant coefficients

EG $y'' + 4y' + 4y = e^{3x}$

$p(D)y = e^{3x}$

need: particular solution $y_p(x)$

$\rightsquigarrow$ general solution: $y_p(x) + y_h(x)$

homogeneous equation:

$y'' + 4y' + 4y = 0$

$p(D)y = 0$

with $p(D) = D^2 + 4D + 4$

roots: $-2, -2$

general solution:

$y_h(x) = e^{-2x}(C_1 + C_2 x)$

"trick" $(D-3)e^{3x} = 0$

$(D-3)p(D)y = 0$

roots: $-2, -2, 3$

general solution:

$e^{-2x}(C_1 + C_2 x) + C_3 e^{3x}$

$\rightsquigarrow$ there must be a particular sol. y_p of the form

$y_p(x) = C_3 e^{3x}$

plug into original DE to find C_3:

$y_p'' + 4y_p' + 4y_p \overset{!}{=} e^{3x}$

$y_p' = 3C_3 e^{3x}$

$y_p'' = 9C_3 e^{3x}$

$9C_3 \cancel{e^{3x}} + 12C_3 \cancel{e^{3x}} + 4C_3 \cancel{e^{3x}} \overset{!}{=} \cancel{e^{3x}}$

$25C_3 = 1 \rightsquigarrow C_3 = \frac{1}{25}$

general solution:

$\underbrace{e^{-2x}(C_1 + C_2 x)}_{y_h} + \underbrace{\frac{1}{25}e^{3x}}_{y_p}$