Example 135. Find the unique solution $u(x,y)$ to:

$$\begin{array}{rl} u_{xx}+u_{yy}=0 & \text{(PDE)} \\ u(x,0)=0 & \\ u(x,2)=3 & \text{(BC)} \\ u(0,y)=0 & \\ u(1,y)=0 & \end{array}$$

Solution. Instead of starting from scratch (homework exercise!), let us reuse our computations:
Let $v(x,y)=u(x,2-y)$. Then $v_{xx}+v_{yy}=0$, $v(x,0)=3$, $v(x,2)=0$, $v(0,y)=0$, $v(1,y)=0$.
Hence, it follows from the previous example that

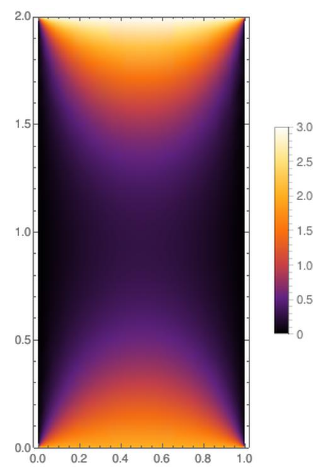
$$v(x,y)=3\sum_{\substack{n=1\\n\text{ odd}}}^{\infty}\frac{4}{\pi n}\frac{1}{1-e^{4\pi n}}\sin(\pi n x)(e^{\pi n y}-e^{-\pi n(y-4)}).$$

Consequently,

$$u(x,y)=v(x,2-y)=3\sum_{\substack{n=1\\n\text{ odd}}}^{\infty}\frac{4}{\pi n}\frac{1}{1-e^{4\pi n}}\sin(\pi n x)(e^{\pi n(2-y)}-e^{\pi n(2+y)}).$$

Example 136. Find the unique solution $u(x,y)$ to:

$$\begin{array}{rr} & u_{xx}+u_{yy} = 0 \\ u(x,0)=2, & u(x,2)=3 \\ u(0,y)=0, & u(1,y)=0 \end{array}$$



Solution. Note that $u(x,y)$ is a combination of the solutions to the previous two examples!

$$u(x,y)=\sum_{\substack{n=1\\n\text{ odd}}}^{\infty}\frac{4}{\pi n}\frac{\sin(\pi n x)}{1-e^{4\pi n}}[2(e^{\pi n y}-e^{-\pi n(y-4)})+3(e^{\pi n(2-y)}-e^{\pi n(2+y)})].$$

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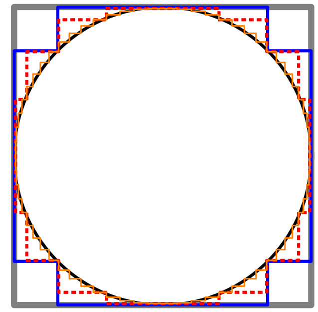
(Just for fun!) π is the perimeter of a circle enclosed in a square with edge length 1. The perimeter of the square is 4, which approximates π. To get a better approximation, we "fold" the vertices of the square towards the circle (and get the blue polygon). This construction can be repeated for even better approximations and, in the limit, our shape will converge to the true circle. At each step, the perimeter is 4, so we conclude that $\pi = 4$, contrary to popular belief.



Can you pin-point the fallacy in this argument?

The solution is below!

($\pi = 4$, solution)

We are constructing curves c_n with the property that $c_n \to c$ where c is the circle. This convergence can be understood, for instance, in the same sense $\|c_n - c\| \to 0$ with the norm measuring the maximum distance between the two curves.

Since $c_n \to c$ we then wanted to conclude that $\text{perimeter}(c_n) \to \text{perimeter}(c)$, leading to $4 \to \pi$.

However, in order to conclude from $x_n \to x$ that $f(x_n) \to f(x)$ we need that f is continuous (at x)!!

The "function" perimeter, however, is not continuous. In words, this means that (as we see in this example) curves can be arbitrarily close, yet have very different arc length.

We can dig a little deeper: as you learned in Calculus II, the arc length of a function $y = f_n(x)$ for $x \in [a, b]$ is

$$\int_a^b \sqrt{(\mathrm{d}x)^2 + (\mathrm{d}y)^2} = \int_a^b \sqrt{1 + f_n'(x)^2} \mathrm{d}x.$$

Observe that this involves $f_n'(x)$. Try to see why the operator D that sends f to f' is not continuous with respect to the distance induced by the norm

$$\|f\| = \left(\int_a^b f(x)^2 \mathrm{d}x \right)^{1/2}.$$

In words, two functions f and g can be arbitrarily close, yet have very different derivatives f' and g'.

That's a huge issue in **functional analysis**, which is the generalization of linear algebra to infinite dimensional spaces (like the space of all differentiable functions). The linear operators ("matrices") on these spaces frequently fail to be continuous.

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