Steady-state temperature

Review. (2D and 3D heat equation) In higher dimensions, the heat equation takes the form $u_t = k(u_{xx} + u_{yy})$ or $u_t = k(u_{xx} + u_{yy} + u_{zz})$.

Note that $\Delta u = u_{xx} + u_{yy} + u_{zz}$ is the Laplace operator you may know from Calculus III (more below).

If temperature is steady, then $u_t = 0$. Hence, the steady-state temperature $u(x, y)$ must satisfy the PDE $u_{xx} + u_{yy} = 0$.

(Laplace equation)

$$u_{xx} + u_{yy} = 0$$

Comment. The Laplace equation is so important that its solutions have their own name: **harmonic functions**.

Comment. Also known as the "potential equation"; satisfied by electric/gravitational potential functions.

Recall from Calculus III (if you have taken that class) that the gradient of a scalar function $f(x, y)$ is the vector field $\boldsymbol{F} = \operatorname{grad} f = \nabla f = \begin{bmatrix} f_x(x, y) \\ f_y(x, y) \end{bmatrix}$. One says that $\boldsymbol{F}$ is a **gradient field** and f is a **potential function** for $\boldsymbol{F}$ (for instance, $\boldsymbol{F}$ could be a gravitational field with gravitational potential f).

The divergence of a vector field $\boldsymbol{G} = \begin{bmatrix} g(x, y) \\ h(x, y) \end{bmatrix}$ is $\operatorname{div} \boldsymbol{G} = g_x + h_y$. One also writes $\operatorname{div} \boldsymbol{G} = \nabla \cdot \boldsymbol{G}$.

The gradient field of a scalar function f is divergence-free if and only if f satisfies the Laplace equation $\Delta f = 0$.

Other notations. $\Delta f = \operatorname{div} \operatorname{grad} f = \nabla \cdot \nabla f = \nabla^2 f$

Boundary conditions. For steady-state temperatures profiles, it is natural to prescribe the temperature on the boundary of a region $R \subseteq \mathbb{R}^2$ (or $R \subseteq \mathbb{R}^3$ in the 3D case).

Comment. Gravitational and electrostatic potentials (not in the vacuum) satisfy the **Poisson equation** $u_{xx} + u_{yy} = f(x, y)$, the inhomogeneous version of the Laplace equation.

(Dirichlet problem)

$u_{xx} + u_{yy} = 0$ within region R

$u(x, y) = f(x, y)$ on boundary of R

In general. A Dirichlet problem consists of a PDE, that needs to hold within a region R, and prescribed values on the boundary of that region ("Dirichlet boundary conditions").

In our next example we solve the Dirichlet problem in the case when R is a rectangle.

Important observation. We are using homogeneous boundary conditions for three of the sides. That is actually no loss of generality.

Indeed, note that in order to solve $\begin{array}{rcl} u_{xx} + u_{yy} = 0 & & \text{(PDE)} \\ u(x, 0) = f_1(x) & & \\ u(x, b) = f_2(x) & & \text{(BC)} \\ u(0, y) = f_3(x) & & \\ u(a, y) = f_4(x) & & \end{array}$ we can solve the four Dirichlet problems

$$\begin{array}{llll} u_{xx} + u_{yy} = 0 & u_{xx} + u_{yy} = 0 & u_{xx} + u_{yy} = 0 & u_{xx} + u_{yy} = 0 \\ u(x, 0) = f_1(x) & u(x, 0) = 0 & u(x, 0) = 0 & u(x, 0) = 0 \\ u(x, b) = 0 & u(x, b) = f_2(x) & u(x, b) = 0 & u(x, b) = 0 \\ u(0, y) = 0 & u(0, y) = 0 & u(0, y) = f_3(x) & u(0, y) = 0 \\ u(a, y) = 0 & u(a, y) = 0 & u(a, y) = 0 & u(a, y) = f_4(x) \end{array}$$

and the sum of the four solutions solves the Dirichlet problem we started with.

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Example 133. Find the unique solution $u(x,y)$ to:

$$\begin{aligned} u_{xx}+u_{yy}&=0 \quad \text{(PDE)}\\ u(x,0)&=f(x)\\ u(x,b)&=0 \quad \text{(BC)}\\ u(0,y)&=0\\ u(a,y)&=0 \end{aligned}$$

Solution.

- We proceed as before and look for solutions $u(x,y)=X(x)Y(y)$ (**separation of variables**).
  Plugging into (PDE), we get $X''(x)Y(y)+X(x)Y''(y)$, and so $\frac{X''(x)}{X(x)}=-\frac{Y''(y)}{Y(y)}=\text{const}=:-\lambda$.
  We thus have $X''+\lambda X=0$ and $Y''-\lambda Y=0$.
- From the last three (BC), we get $X(0)=0$, $X(a)=0$, $Y(b)=0$.
  We ignore the first (inhomogeneous) condition for now.
- So X solves $X''+\lambda X=0$, $X(0)=0$, $X(a)=0$.
  From earlier, we know that, up to multiples, the only nonzero solutions of this eigenvalue problem are $X(x)=\sin\left(\frac{\pi n}{a}x\right)$ corresponding to $\lambda=\left(\frac{\pi n}{a}\right)^2$, $n=1,2,3\ldots$.
- On the other hand, Y solves $Y''-\lambda Y=0$, and hence $Y(y)=Ae^{\sqrt{\lambda}y}+Be^{-\sqrt{\lambda}y}$.
  The condition $Y(b)=0$ implies that $Ae^{\sqrt{\lambda}b}+Be^{-\sqrt{\lambda}b}=0$ so that $B=-Ae^{2\sqrt{\lambda}b}$.
  Hence, $Y(y)=A\left(e^{\sqrt{\lambda}y}-e^{-\sqrt{\lambda}(y-2b)}\right)$.
- Taken together, we have the solutions $u_n(x,y)=\sin\left(\frac{\pi n}{a}x\right)\left(e^{\frac{\pi n}{a}y}-e^{-\frac{\pi n}{a}(y-2b)}\right)$ solving (PDE)+(BC), with the exception of $u(x,0)=f(x)$.
- We wish to combine these in such a way that $u(x,0)=f(x)$ holds as well.
  At $y=0$, $u_n(x,0)=\sin\left(\frac{\pi n}{a}x\right)(1-e^{2\pi nb/a})$. All of these are $2a$-periodic.
  Hence, we extend $f(x)$, which is only given on $(0,a)$, to an odd $2a$-periodic function (its Fourier sine series!). By making it odd, its Fourier series will only involve sine terms: $f(x)=\sum_{n=1}^{\infty}b_n\sin\left(\frac{\pi n}{a}x\right)$.
  Note that

  $$b_n=\frac{1}{a}\int_{-a}^{a}f(x)\sin\left(\frac{n\pi x}{a}\right)\mathrm{d}x=\frac{2}{a}\int_0^a f(x)\sin\left(\frac{n\pi x}{a}\right)\mathrm{d}x,$$

  where the first integral makes reference to the extension of $f(x)$ while the second integral only uses $f(x)$ on its original interval of definition.

Consequently, (PDE)+(BC)+(IC) is solved by

$$u(x,y)=\sum_{n=1}^{\infty}\frac{b_n}{1-e^{2\pi nb/a}}u_n(x,y)=\sum_{n=1}^{\infty}\frac{b_n}{1-e^{2\pi nb/a}}\sin\left(\frac{\pi n}{a}x\right)\left(e^{\frac{\pi n}{a}y}-e^{-\frac{\pi n}{a}(y-2b)}\right),$$

where

$$b_n=\frac{2}{a}\int_0^a f(x)\sin\left(\frac{n\pi x}{a}\right)\mathrm{d}x.$$

Example 134. Find the unique solution $u(x,y)$ to:

$$u_{xx} + u_{yy} = 0 \quad \text{(PDE)}$$

$$\begin{array}{r} u(x,0) = 1 \\ u(x,2) = 0 \\ u(0,y) = 0 \\ u(1,y) = 0 \end{array} \quad \text{(BC)}$$

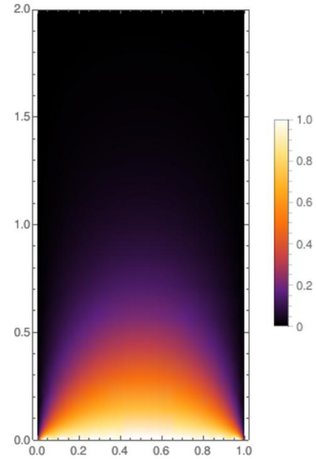
Solution. This is the special case of the previous example with $a = 1$, $b = 2$ and $f(x) = 1$ for $x \in (0,1)$.

From Example 111, we know that $f(x)$ has the Fourier sine series

$$f(x) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{\pi n} \sin(n\pi x), \quad x \in (0,1).$$

Hence,

$$u(x,y) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{\pi n} \frac{1}{1 - e^{4\pi n}} \sin(\pi n x)(e^{\pi n y} - e^{-\pi n (y-4)}).$$



Comment. The temperature at the center is $u(\frac{1}{2}, 1) \approx 0.0549$ (only the first term of the infinite sum suffices for this estimate; the first three terms suffice for 9 digits of accuracy).

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