Review. The heat equation: $u_t = k u_{xx}$

Let us think about what is needed to describe a unique solution of the heat equation.

- **Initial condition** at $t = 0$: $u(x, 0) = f(x)$ (IC)

  This specifies an initial temperature distribution at time $t = 0$.

- **Boundary condition** at $x = 0$ and $x = L$: (BC)

  Assuming that heat only enters/exits at the boundary (think of our rod as being insulated, except possibly at the two ends), we need some condition on the temperature at the ends. For instance:

  - $u(0, t) = A$, $u(L, t) = B$

    This models a rod where one end is kept at temperature A and the other end at temperature B.

  - $u_x(0, t) = u_x(L, t) = 0$

    This models a rod whose ends are insulated as well.

Under such assumptions, our physical intuition suggests that there should be a unique solution.

Important comment. We can always transform the case $u(0, t) = A$, $u(L, t) = B$ into $u(0, t) = u(L, t) = 0$ by using the fact that $u(t, x) = ax + b$ solves $u_t = k u_{xx}$. Can you spell this out?

Example 128. (cont'd) To get a feeling, let us find some solutions to $u_t = u_{xx}$.

- $u(x, t) = ax + b$ is a solution.
- For instance, $u(x, t) = e^t e^x$ is a solution.
  [Not a very interesting one for modeling heat flow because it increases exponentially in time.]
- More interesting are $u(x, t) = e^{-t}\cos(x)$ and $u(x, t) = e^{-t}\sin(x)$.
- More generally, $e^{-n^2 t}\cos(nx)$ and $e^{-n^2 t}\sin(nx)$ are solutions.

Important observation. This actually reveals a strategy for solving the PDE $u_t = u_{xx}$ with conditions such as:

$$u(0, t) = u(\pi, t) = 0 \qquad \text{(BC)}$$
$$u(x, 0) = f(x), \quad x \in (0, L) \qquad \text{(IC)}$$

Namely, the solutions $u_n(x, t) = e^{-n^2 t}\sin(nx)$ all satisfy (BC).
It remains to satisfy (IC). Note that $u_n(x, 0) = \sin(nx)$. To find $u(x, t)$ such that $u(x, 0) = f(x)$, we can write $f(x)$ as a Fourier sine series (i.e. extend $f(x)$ to a 2π-periodic odd function):

$$f(x) = \sum_{n \geqslant 1} b_n \sin(nx)$$

Then $u(x, t) = \sum_{n \geqslant 1} b_n u_n(x, t) = \sum_{n \geqslant 1} b_n e^{-n^2 t}\sin(nx)$ solves the PDE $u_t = u_{xx}$ with (BC) and (IC).

Example 129. Find the unique solution to:

$$\begin{aligned} u_t &= k u_{xx} && \text{(PDE)} \\ u(0,t) &= u(L,t) = 0 && \text{(BC)} \\ u(x,0) &= f(x), \quad x \in (0,L) && \text{(IC)} \end{aligned}$$

Solution.

- We will first look for simple solutions of (PDE)+(BC) (and then we plan to take a combination of such solutions that satisfies (IC) as well). Namely, we look for solutions $u(x,t) = X(x)T(t)$. This approach is called **separation of variables** and it is crucial for solving other PDEs as well.
- Plugging into (PDE), we get $X(x)T'(t) = kX''(x)T(t)$, and so $\frac{X''(x)}{X(x)} = \frac{T'(t)}{kT(t)}$.

  Note that the two sides cannot depend on x (because the right-hand side doesn't) and they cannot depend on t (because the left-hand side doesn't). Hence, they have to be constant. Let's call this constant $-\lambda$.

  Then, $\frac{X''(x)}{X(x)} = \frac{T'(t)}{kT(t)} = \text{const} =: -\lambda$.

  We thus have $X'' + \lambda X = 0$ and $T' + \lambda k T = 0$.
- Consider (BC). Note that $u(0,t) = X(0)T(t) = 0$ implies $X(0) = 0$.

  [Because otherwise $T(t) = 0$ for all t, which would mean that $u(x,t)$ is the dull zero solution.]

  Likewise, $u(L,t) = X(L)T(t) = 0$ implies $X(L) = 0$.
- So X solves $X'' + \lambda X = 0$, $X(0) = 0$, $X(L) = 0$. We know that, up to multiples, the only nonzero solutions are the eigenfunctions $X(x) = \sin\left(\frac{\pi n}{L} x\right)$ corresponding to the eigenvalues $\lambda = \left(\frac{\pi n}{L}\right)^2$, $n = 1, 2, 3 \ldots$
- On the other hand, T solves $T' + \lambda k T = 0$, and hence $T(t) = e^{-\lambda k t} = e^{-\left(\frac{\pi n}{L}\right)^2 k t}$.
- Taken together, we have the solutions $u_n(x,t) = e^{-\left(\frac{\pi n}{L}\right)^2 k t} \sin\left(\frac{\pi n}{L} x\right)$ solving (PDE)+(BC).
- We wish to combine these in such a way that (IC) holds as well.

  At $t = 0$, $u_n(x,0) = \sin\left(\frac{\pi n}{L} x\right)$. All of these are $2L$-periodic.

  Hence, we extend $f(x)$, which is only given on $(0, L)$, to an odd $2L$-periodic function (its Fourier sine series!). By making it odd, its Fourier series will only involve sine terms: $f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{\pi n}{L} x\right)$.

Consequently, (PDE)+(BC)+(IC) is solved by

$$u(x,t) = \sum_{n=1}^{\infty} b_n u_n(x,t) = \sum_{n=1}^{\infty} b_n e^{-\left(\frac{\pi n}{L}\right)^2 k t} \sin\left(\frac{\pi n}{L} x\right).$$

Example 130. Find the unique solution to:

$$\begin{aligned} u_t &= u_{xx} \\ u(0,t) &= u(1,t) = 0 \\ u(x,0) &= 1, \quad x \in (0,1) \end{aligned}$$

Solution. This is the case $k = 1$, $L = 1$ and $f(x) = 1$, $x \in (0,1)$, of the previous example.

In the final step, we extend $f(x)$ to the 2-periodic odd function of Example 111. In particular, earlier, we have already computed that the Fourier series is

$$f(x) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{\pi n} \sin(n\pi x).$$

Hence, $u(x,t) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{\pi n} e^{-\pi^2 n^2 t} \sin(n\pi x)$.

Comment. Note that, for $t > 0$, the exponential very quickly approaches 0 (because of the $-n^2$ in the exponent), so that we get very accurate approximations with only a handful terms.

Make some 3D plots!