Notes for Lecture 1

Mon, 1/13/2025

Review: Computing derivatives

Given a function $y(x)$, we learned in Calculus I that its **derivative**

$$y'(x) = \frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

(where $\Delta y = y(x + \Delta x) - y(x)$) has the following two important characterizations:

- $y'(x)$ is the **slope of the tangent line** of the graph of $y(x)$ at x, and
- $y'(x)$ is the **rate of change** of $y(x)$ at x.

Comment. Derivatives were introduced in the late 1600s by Newton and Leibniz who later each claimed priority in laying the foundations for calculus. Certainly both of them contributed mightily to those foundations.

Moreover, we learned simple rules to compute the derivative of functions:

- **(sum rule)** $\frac{\mathrm{d}}{\mathrm{d}x}[f(x) + g(x)] = f'(x) + g'(x)$
- **(product rule)** $\frac{\mathrm{d}}{\mathrm{d}x}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$
- **(chain rule)** $\frac{\mathrm{d}}{\mathrm{d}x}[f(g(x))] = f'(g(x))g'(x)$

  Comment. If we write $t = g(x)$ and $y = f(t)$, then the chain rule takes the form $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t} \cdot \frac{\mathrm{d}t}{\mathrm{d}x}$.
  In other words, the chain rule expresses the fact that we can treat $\frac{\mathrm{d}y}{\mathrm{d}x}$ (which initially is just a notation for $y'(x)$) as an honest fraction.

- **(basic functions)** $\frac{\mathrm{d}}{\mathrm{d}x} x^r = r x^{r-1}$,
  $\frac{\mathrm{d}}{\mathrm{d}x} e^x = e^x$, $\frac{\mathrm{d}}{\mathrm{d}x} \ln(x) = \frac{1}{x}$,
  $\frac{\mathrm{d}}{\mathrm{d}x} \sin(x) = \cos(x)$, $\frac{\mathrm{d}}{\mathrm{d}x} \cos(x) = -\sin(x)$

These rules are enough to compute the derivative of any function that we can build from the basic functions using algebraic operations and composition. On the other hand, as you probably recall from Calculus II, reversing the operation of differentiation (i.e. computing antiderivatives) is much more difficult.

In particular, there exist simple functions (such as e^{x^2}) whose antiderivative cannot be expressed in terms of the basic functions above.

Example 1. Derive the **quotient rule** from the rules above.

Solution. We write $\frac{f(x)}{g(x)} = f(x) \cdot \frac{1}{g(x)}$ and apply the product rule to get

$$\frac{\mathrm{d}}{\mathrm{d}x} f(x) \cdot \frac{1}{g(x)} = f'(x)\frac{1}{g(x)} + f(x)\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{g(x)}.$$

By the chain rule combined with $\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{x} = -\frac{1}{x^2}$, we have $\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{g(x)} = -\frac{1}{g(x)^2}g'(x)$. Using this in the previous formula,

$$\frac{\mathrm{d}}{\mathrm{d}x} f(x) \cdot \frac{1}{g(x)} = f'(x)\frac{1}{g(x)} - f(x)\frac{1}{g(x)^2}g'(x) = \frac{f'(x)}{g(x)} - \frac{f(x)g'(x)}{g(x)^2}.$$

Putting the final two fractions on a common denominator, we obtain the familiar quotient rule

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{f(x)}{g(x)} = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$$

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Example 2. Compute the following derivatives:

(a) $\frac{\mathrm{d}}{\mathrm{d}x}(5x^3+7x^2+2)$

(b) $\frac{\mathrm{d}}{\mathrm{d}x}\sin(5x^3+7x^2+2)$

(c) $\frac{\mathrm{d}}{\mathrm{d}x}(x^3+2x)\sin(5x^3+7x^2+2)$

Solution.

(a) $\frac{\mathrm{d}}{\mathrm{d}x}(5x^3+7x^2+2) = 15x^2+14x$

(b) $\frac{\mathrm{d}}{\mathrm{d}x}\sin(5x^3+7x^2+2) = (15x^2+14x)\cos(5x^3+7x^2+2)$

(c) $\frac{\mathrm{d}}{\mathrm{d}x}(x^3+2x)\sin(5x^3+7x^2+2)$
$= (3x^2+2)\sin(5x^3+7x^2+2) + (x^3+2x)(15x^2+14x)\cos(5x^3+7x^2+2)$

First examples of differential equations

Example 3. Here are two first examples of a **differential equation** (DE):

(a) $y' = 2xy$

This is short for $y'(x) = 2xy(x)$. The goal is to find a function $y(x)$ satisfying this equation.

One such **solution** is $y(x) = e^{x^2}$. We will soon learn techniques to find this ourselves but, already now, we can verify that it is indeed a solution: if $y(x) = e^{x^2}$ then $y'(x) = 2xe^{x^2} = 2xy(x)$.

(b) $(xy' - 3y''')^2 = 5\sin(2x+y^4)+7$

This illustrates that y and its derivative can show up in any kind of way. We say that this DE has **order** 3 because the highest derivative is the 3rd derivative y'''.

Example 4. Solve the DE $y' = x^2+x$.

Solution. Note that the DE simply asks for a function $y(x)$ with a specific derivative (in particular, the right-hand side does not involve $y(x)$). In other words, the desired $y(x)$ is an **antiderivative** of x^2+x. We know from Calculus II that we can find antiderivatives by integrating:

$$y(x) = \int (x^2+x)\mathrm{d}x = \frac{1}{3}x^3 + \frac{1}{2}x^2 + C$$

Moreover, we know from Calculus II that there are no other solutions. In other words, we found the **general solution** to the DE.

Example 5. As in the previous example, any DE of the form $y' = f(x)$ (this is artificially easy) is just asking us to compute an antiderivative of $f(x)$.

On the other hand, this is an early indication that solving DEs is hard (and includes computing integrals as a special case). For instance, the DE $y' = e^{x^2}$ requires us to compute the antiderivative of e^{x^2}. It turns out that this cannot be done using the basic functions we know from Calculus.

Advanced comment. A "solution" to the above issue is to **define** a new function as the antiderivative that we presently cannot write down a formula for. Look up the so-called **error function** if you are curious!

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Example 6. (review) Solve the DE $y' = x^2 + x$.

Solution. Note that the DE simply asks for a function $y(x)$ with a specific derivative (in particular, the right-hand side does not involve $y(x)$). In other words, the desired $y(x)$ is an **antiderivative** of $x^2 + x$. We know from Calculus II that we can find antiderivatives by integrating:

$$y(x) = \int (x^2 + x)\mathrm{d}x = \frac{1}{3}x^3 + \frac{1}{2}x^2 + C$$

Moreover, we know from Calculus II that there are no other solutions. In other words, we found the **general solution** to the DE.

If the highest derivative appearing in a DE is an rth derivative, we say that the DE has **order** r.

For instance. The DE $y' = 3\sqrt{1 - y^2}$ has order 1 (such DEs are also called first order DEs).
On the other hand, the DE $y'' = y' + 6y$ has order 2 (such DEs are also called second order DEs).

As we will observe in the next few examples, we typically expect that the general solution of a DE of **order** r has r **parameters** (or degrees of freedom).

A first initial value problem

To single out a **particular solution**, we need to specify additional conditions (typically one condition per parameter in the general solution). For instance, it is common to impose **initial conditions** such as $y(1) = 2$. A DE together with an initial condition is called an **initial value problem** (IVP).

Example 7. Solve the IVP $y' = x^2 + x$ with $y(1) = 2$.

Solution. From the previous example, we know that $y(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 + C$.
Since $y(1) = \frac{1}{3} + \frac{1}{2} + C = \frac{5}{6} + C \overset{!}{=} 2$, we find $C = 2 - \frac{5}{6} = \frac{7}{6}$.
Hence, $y(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 + \frac{7}{6}$ is the (unique) solution of the IVP.

Example 8. (homework) Solve the DE $y'' = x^2 + x$.

Solution. We now take two antiderivatives of $x^2 + x$ to get

$$y(x) = \int\int (x^2 + x)\mathrm{d}x\mathrm{d}x = \int \left(\frac{1}{3}x^3 + \frac{1}{2}x^2 + C\right)\mathrm{d}x = \frac{1}{12}x^4 + \frac{1}{6}x^3 + Cx + D,$$

where it is important that we give the second constant of integration a name different from the first.
Important comment. This is the general solution to the DE. The DE is of order 2 and, as expected, the general solution has 2 parameters.

Verifying if a function solves a DE

Given a function, we can always check whether it solves a DE!

We can just plug it into the DE and see if left and right side agree. This means that we can always check our work as well as that we can verify solutions generated by someone else (or a computer algebra system) even if we don't know the techniques for solving the DE.

Example 9. (warmup) Consider the DE $y'' = y' + 6y$.

(a) Is $y(x) = e^{2x}$ a solution?

(b) Is $y(x) = e^{3x}$ a solution?

Solution.

(a) We compute $y' = 2e^{2x}$ and $y'' = 4e^{2x}$.
Since $y' + 6y = 8e^{2x}$ is different from $y'' = 4e^{2x}$, we conclude that $y(x) = e^{2x}$ is not a solution.

(b) We compute $y' = 3e^{3x}$ and $y'' = 9e^{3x}$.
Since $y' + 6y = 9e^{3x}$ is equal to $y'' = 9e^{3x}$, we conclude that $y(x) = e^{3x}$ is a solution of the DE.

We will soon be able to completely solve differential equations such as in the previous example. The following gives a taste of how we will go about it:

Example 10. (cont'd) Consider the DE $y'' = y' + 6y$. For which r is e^{rx} a solution?

Solution. If $y(x) = e^{rx}$, then $y'(x) = re^{rx}$ and $y''(x) = r^2 e^{rx}$.
Plugging $y(x) = e^{rx}$ into the DE, we get $r^2 e^{rx} = re^{rx} + 6e^{rx}$ which simplifies to $r^2 = r + 6$.
This has the two solutions $r = -2$, $r = 3$. Hence e^{-2x} and e^{3x} are solutions of the DE.

In fact, we check that $Ae^{-2x} + Be^{3x}$ is a **two-parameter family** of solutions to the DE.

Important comment. It is no coincidence that the order of the DE is 2, whereas the previous example has order 1. In general, we expect a DE of order r to have a solution with r parameters.

Example 11. Consider the DE $e^y y' = 1$.

(a) Is $y(x) = \ln(x)$ a solution to the DE?

(b) Is $y(x) = \ln(x) + C$ a solution to the DE?

(c) Is $y(x) = \ln(x + C)$ a solution to the DE?

Solution.

(a) Since $y'(x) = \frac{1}{x}$ and $e^{y(x)} = e^{\ln(x)} = x$, we have $e^y y' = x \cdot \frac{1}{x} \overset{\checkmark}{=} 1$.
Hence, $y(x) = \ln(x)$ is a solution to the given DE.

(b) Since $y'(x) = \frac{1}{x}$ and $e^{y(x)} = e^{\ln(x) + C} = xe^C$, we have $e^y y' = xe^C \cdot \frac{1}{x} = e^C$. Thus the DE is satisfied only if $e^C = 1$ which only happens if $C = 0$ (which is the case in the first part).
Hence, $y(x) = \ln(x) + C$ is not a solution to the given DE except if $C = 0$.

(c) Since $y'(x) = \frac{1}{x + C}$ and $e^{y(x)} = e^{\ln(x + C)} = x + C$, we have $e^y y' = (x + C) \cdot \frac{1}{x + C} \overset{\checkmark}{=} 1$.
Hence, $y(x) = \ln(x + C)$ is indeed a one-parameter family of solutions to the given DE.

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Any fixed function solves many DEs

Usually, we start with a DE (which comes, for instance, from physical laws) and want to solve it. In the next example, we start with a function and determine several DEs that it solves.

Example 12. Determine several (random) DEs that $y(x) = \sin(3x)$ solves.

Solution. Here are some options (but there are many more):

(a) We compute $y'(x) = 3\cos(3x)$. Accordingly, $y(x) = \sin(3x)$ solves the DE $y' = 3\cos(3x)$.

Comment. This, however, is not an "interesting" choice. In particular, this DE could be simply solved by computing an antiderivative (as in the previous examples).

Comment. Note that there are further solutions to this DE: the **general solution** is $\int 3\cos(3x)\mathrm{d}x = \sin(3x) + C$ where C is any constant. We say that $y(x) = \sin(3x) + C$ is a **one-parameter family** of solutions to the DE. C is called a **degree of freedom**.

(b) Note that $y'(x) = 3\cos(3x) = 3\sqrt{1 - (\sin(3x))^2} = 3\sqrt{1 - y(x)^2}$ (for x close to 0).

[Here we used that $\cos(x)^2 + \sin(x)^2 = 1$, which implies that $\cos(x) = \sqrt{1 - \sin(x)^2}$.]

Hence, $y(x) = \sin(3x)$ solves the differential equation $y' = 3\sqrt{1 - y^2}$.

Comment. In the above, we restrict x to $\left(-\frac{\pi}{6}, \frac{\pi}{6}\right)$ so that $\cos(3x) > 0$. Less precisely, we can say that x is close to 0. (It is a common feature of DEs that we work with values of x close to a certain initial value.)

(c) We compute $y''(x) = -9\sin(3x)$. Accordingly, $y(x) = \sin(3x)$ solves the DE $y'' = -9\sin(3x)$.

Comment. Once more this DE is easy (because it only involves y'' but not y or y'). Hence, we can find the general solution by simply taking two antiderivatives:

$$y(x) = \iint -9\sin(3x)\mathrm{d}x\,\mathrm{d}x = \int (3\cos(3x) + C)\mathrm{d}x = \sin(3x) + Cx + D.$$

It is important that we give the second constant of integration a name different from the first. That way, we see that the general solution has 2 degrees of freedom. This matches the fact that the order of the DE is 2.

Important comment. This is no coincidence. In general, we expect a DE of order r to have a general solution with r parameters.

(d) $y(x) = \sin(3x)$ also solves the DE $y'' = -9y$.

Comment. This is again a DE of order 2. Therefore the general solution should have 2 degrees of freedom. Later we will learn to solve such DEs. For now, we can verify that $y(x) = A\sin(3x) + B\cos(3x)$ is a solution for any constants A and B.

Homework. Check that $y(x) = \sin(3x) + C$ does not solve the DE $y'' = -9y$.

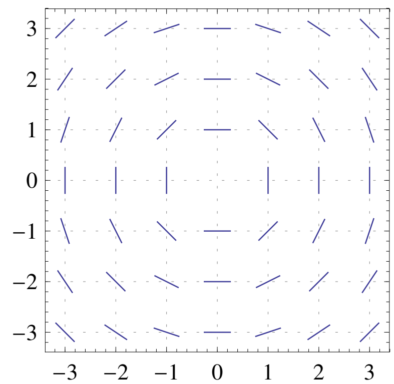
Slope fields, or sketching solutions to DEs

The next example illustrates that we can "plot" solutions to differential equations (it does not matter if we are able to actually solve them).

Comment. This is an important point because "plotting" really means that we can numerically approximate solutions. For complicated systems of differential equations, such as those used to model fluid flow, this is usually the best we can do. Nobody can actually solve these equations.

Example 13. Consider the DE $y' = -x/y$.

Let's pick a point, say, $(1,2)$. If a solution $y(x)$ is passing through that point, then its slope has to be $y' = -1/2$. We therefore draw a small line through the point $(1,2)$ with slope $-1/2$. Continuing in this fashion for several other points, we obtain the **slope field** on the right.



With just a little bit of imagination, we can now anticipate the solutions to look like (half)circles around the origin. Let us check whether $y(x) = \sqrt{r^2 - x^2}$ might indeed be a solution!

$y'(x) = \frac{1}{2}\frac{-2x}{\sqrt{r^2 - x^2}} = -x/y(x)$. So, yes, we actually found solutions!

Solving DEs: Separation of variables

Example 14. Solve the DE $y' = -\frac{x}{y}$.

Solution. Rewrite the DE as $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}$.

Separate the variables to get $y\,\mathrm{d}y = -x\,\mathrm{d}x$ (in particular, we are multiplying both sides by $\mathrm{d}x$).

Integrating both sides, we get $\int y\,\mathrm{d}y = \int -x\,\mathrm{d}x$.

Computing both integrals results in $\frac{1}{2}y^2 = -\frac{1}{2}x^2 + C$ (we combine the two constants of integration into one).

Hence $x^2 + y^2 = D$ (with $D = 2C$).

This is an **implicit form** of the solutions to the DE. We can make it explicit by solving for y. Doing so, we find $y(x) = \pm\sqrt{D - x^2}$ (choosing $+$ gives us the upper half of a circle, while the negative sign gives us the lower half).

Comment. The step above where we break $\frac{\mathrm{d}y}{\mathrm{d}x}$ apart and then integrate may sound sketchy!

However, keep in mind that, after we find a solution $y(x)$, even if by sketchy means, we can (and should!) verify that $y(x)$ is indeed a solution by plugging into the DE. We actually already did that in the previous example!

Example 15. Solve the IVP $y' = -\frac{x}{y}$, $y(0) = -3$.

Comment. Instead of using what we found in Example 14, we start from scratch to better illustrate the solution process (and how to use the initial condition right away to determine the value of the constant of integration).

Solution. We separate variables to get $y\,\mathrm{d}y = -x\,\mathrm{d}x$.

Integrating gives $\frac{1}{2}y^2 = -\frac{1}{2}x^2 + C$, and we use $y(0) = -3$ to find $\frac{1}{2}(-3)^2 = 0 + C$ so that $C = \frac{9}{2}$.

Hence, $x^2 + y^2 = 9$ is an **implicit** form of the solution.

Solving for y, we get $y = -\sqrt{9 - x^2}$ (note that we have to choose the negative sign so that $y(0) = -3$).

Comment. Note that our solution is a **local solution**, meaning that it is valid (and solves the DE) locally around $x = 0$ (from the initial condition). However, it is not a **global solution** because it doesn't make sense outside of x in the interval $[-3, 3]$.

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Which differential equations can we actually solve using separation of variables?

- A general DE of first order is typically of the form $\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y)$.

  For instance, $\frac{\mathrm{d}y}{\mathrm{d}x} = \sin(xy) - x^2 y$.

  Comment. First order means that only the first derivative of y shows up. The most general form of a DE of first order is $F(x, y, y') = 0$ but we can usually solve for y' to get to the above form.

- The ones we can solve are **separable equations**, which are of the form $\frac{\mathrm{d}y}{\mathrm{d}x} = g(x)h(y)$.

  Example. The equation $\frac{\mathrm{d}y}{\mathrm{d}x} = y - x$ (although simple) is not separable.

  Example. The equation $\frac{\mathrm{d}y}{\mathrm{d}x} = e^{y-x}$ is separable because we can write it as $\frac{\mathrm{d}y}{\mathrm{d}x} = e^y e^{-x}$.

Example 16. (extra)

Comment. In this example, we use $x(t)$ instead of $y(x)$ for the function described by the differential equation. In general, of course, any choice of variable names is possible. If we write something like x' or y' it needs to be clear from the context with respect to which variable that derivative is meant (such as $x' = \frac{\mathrm{d}}{\mathrm{d}t}x(t)$).

(a) Solve the DE $\frac{\mathrm{d}x}{\mathrm{d}t} = kx^2$.

(b) Verify your answer from the first part.

(c) Solve the IVP $\frac{\mathrm{d}x}{\mathrm{d}t} = kx^2$, $x(0) = 2$.

(d) Solve the IVP $\frac{\mathrm{d}x}{\mathrm{d}t} = kx^2$, $x(0) = 0$.

Solution.

(a) This DE is separable: $\frac{1}{x^2}\mathrm{d}x = k\,\mathrm{d}t$. Integrating, we find $-\frac{1}{x} = kt + B$. (We plan to replace B by a new constant C in a moment.) Hence, $x = -\frac{1}{kt + B} = \frac{1}{C - kt}$.

[Here, $C = -B$ but that relationship doesn't matter; it only matters that the solution has a free parameter.]

Comment. Note that we did not find the solution $x = 0$ (lost when dividing by x^2). It is called a **singular solution** because it is not part of the **general solution** (the one-parameter family found above). [Although, we can obtain it from the general solution by letting $C \to \infty$.]

See the last part for a case when this "missing" solution is needed.

(b) Starting with $x(t) = \frac{1}{C - kt}$, we compute that $\frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{1}{(C - kt)^2} \cdot (-k) = \frac{k}{(C - kt)^2}$.

On the other hand, $kx^2 = k\left(\frac{1}{C - kt}\right)^2 = \frac{k}{(C - kt)^2}$. Since this matches what we got for $\frac{\mathrm{d}x}{\mathrm{d}t}$, it is indeed true that $\frac{\mathrm{d}x}{\mathrm{d}t} = kx^2$.

(c) We start with $x(t) = \frac{1}{C - kt}$ (which we know solves the DE for any value of C) and seek to choose C so that $x(0) = 2$.

Since $x(0) = \left[\frac{1}{C - kt}\right]_{t=0} = \frac{1}{C} \overset{!}{=} 2$, we find $C = \frac{1}{2}$.

Hence, the IVP has the (unique) solution $x(t) = \frac{1}{1/2 - kt}$.

(d) Proceeding as in the previous part, we now arrive at the impossible equation $\frac{1}{C} \overset{!}{=} 0$.

However, this suggests that we should consider taking $C \to \infty$ in $x(t) = \frac{1}{C - kt}$, which results in $x(t) = 0$.

Indeed, it is easy to verify (make sure you know what this entails!) that $x(t) = 0$ solves the IVP.

In the following example, we first proceed like we did when producing a slope field to compute slopes (and, therefore, tangent lines) of solutions. Indeed, besides the slope y', we can further compute further derivatives like y'' or y''' by differentiating the DE.

Do you recall how y'' tells us about the curvature of a function $y(x)$?

Example 17. Consider the DE $x^2y' = 1 + xy^3$. Suppose that $y(x)$ is a solution passing through the point $(2, 1)$.

Important. This is the same as saying that $y(x)$ solves the IVP $x^2y' = 1 + xy^3$ with $y(2) = 1$.

(a) Determine $y'(2)$.

(b) Determine the tangent line of $y(x)$ at $(2, 1)$.

(c) Determine $y''(2)$.

Comment. Note that this DE is not separable.

Solution.

(a) At the point $(2, 1)$ we have $x = 2$ and $y = 1$. Plugging these values into the differential equation, we get $4y' = 1 + 2 \cdot 1^3 = 3$ which we can solve for y' to find $y' = \frac{3}{4}$.

Since y' is short for $y'(x) = y'(2)$, we have found $y'(2) = \frac{3}{4}$.

(b) The tangent line is the line through $(2, 1)$ with slope $\frac{3}{4}$ (computed in the previous part).

From this information, we can immediately write down its equation in the form $y = \frac{3}{4}(x - 2) + 1$.

(c) To get our hands on $y''(2)$, we can differentiate (with respect to x) both sides of $x^2y' = 1 + xy^3$. Applying the product rule, we have $\frac{\mathrm{d}}{\mathrm{d}x}x^2y'(x) = 2xy'(x) + x^2y''(x) = 2xy' + x^2y''$ as well as $\frac{\mathrm{d}}{\mathrm{d}x}(1 + xy(x)^3) = y(x)^3 + x \cdot 3y(x)^2 \cdot y'(x) = y^3 + 3xy^2y'$.
Thus $2xy' + x^2y'' = y^3 + 3xy^2y'$. To find $y''(2)$, we plug in $x = 2$, $y = 1$, $y' = \frac{3}{4}$.

This results in $2 \cdot 2 \cdot \frac{3}{4} + 4y'' = 1 + 3 \cdot 2 \cdot 1 \cdot \frac{3}{4}$ or $3 + 4y'' = \frac{11}{2}$. It follows that $y'' = \frac{1}{4} \cdot \frac{5}{2} = \frac{5}{8}$.

Comment. Alternatively, we can rewrite the DE as $y' = \frac{1}{x^2} + \frac{1}{x}y^3$ and then differentiate. Do it!

Comment. Do you recall from Calculus what it means visually to have $y'' = \frac{5}{8}$?

[Since $y'' > 0$ it means that our function is concave up at $(2, 1)$. As such, its graph will lie above the tangent line.]

Comment. Note that we could continue and likewise find $y'''(2)$ or higher derivatives at $x = 2$. This is the starting point for the power series method typically discussed in Differential Equations II.

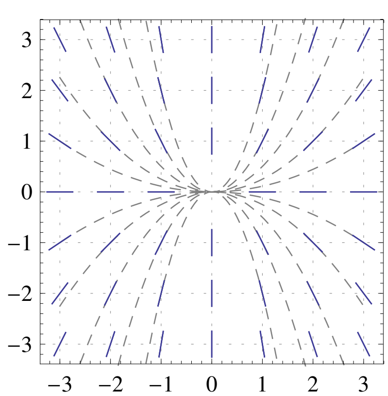
Example 18. (warm-up) Consider the DE $xy' = 2y$.

Sketch its slope field.

Challenge. Try to guess solutions $y(x)$ from the slope field.

Solution. For instance, to find the slope at the point $(3, 1)$, we plug $x = 3$, $y = 1$ into the DE to get $3y' = 2$. Hence, the slope is $y' = 2/3$.

The resulting slope field is sketched on the right.



Solution of the challenge. Trace out the solution through $(1, 1)$ (and then some other points). Their shape looks like a parabola, so that we might guess that $y(x) = Cx^2$ solves the DE. Check that this is indeed the case by plugging into the DE!

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Solving DEs: Separation of variables, cont'd

In general, **separation of variables** solves $y' = g(x)h(y)$ by writing the DE as $\frac{1}{h(y)}\mathrm{d}y = g(x)\mathrm{d}x$.

Note that $\frac{1}{h(y)}\frac{\mathrm{d}y}{\mathrm{d}x} = g(x)$ is indeed equivalent to $\int \frac{1}{h(y)}\mathrm{d}y = \int g(x)\mathrm{d}x + C$. Why?! (Apply $\frac{\mathrm{d}}{\mathrm{d}x}$ to the integrals...)

Example 19. (cont'd) Solve the IVP $xy' = 2y$, $y(1) = 2$.

Solution. Rewrite the DE as $\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2}{x}$.

Then multiply both sides with $\mathrm{d}x$ and integrate both of them to get $\int \frac{1}{y}\mathrm{d}y = \int \frac{2}{x}\mathrm{d}x$.

Hence, $\ln|y| = 2\ln|x| + C$.

The initial condition $y(1) = 2$ tells us that, at least locally, $x > 0$ and $y > 0$. Thus $\ln(y) = 2\ln(x) + C$.

Moreover, plugging in $x = 1$ and $y = 2$, we find $C = \ln(2)$.

Solving $\ln(y) = 2\ln(x) + \ln(2)$ for y, we find $y = e^{2\ln(x) + \ln(2)} = 2x^2$.

Comment. When solving a DE or IVP, we can generally only expect to find a **local solution**, meaning that our solution might only be valid in a small interval around the initial condition (here, we can only expect $y(x)$ to be a solution for all x in an interval around 1; especially since we assumed $x > 0$ in our solution). However, we can check (do it!) that the solution $y = 2x^2$ is actually a **global solution** (meaning that it is a solution for all x, not just locally around 1).

Let's solve the same differential equation with a different choice of initial condition:

Example 20. Solve the IVP $xy' = 2y$, $y(1) = -1$.

Solution. Again, we rewrite the DE as $\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2}{x}$, multiply both sides with $\mathrm{d}x$, and integrate to get $\int \frac{1}{y}\mathrm{d}y = \int \frac{2}{x}\mathrm{d}x$.

Hence, $\ln|y| = 2\ln|x| + C$. The initial condition $y(1) = -1$ tells us that, at least locally, $x > 0$ and $y < 0$ (note that this means $|y| = -y$). Thus $\ln(-y) = 2\ln(x) + C$.

Moreover, plugging in $x = 1$ and $y = -1$, we find $C = 0$.

Solving $\ln(-y) = 2\ln(x)$ for y, we find $y = -e^{2\ln(x)} = -x^2$. We easily verify that this is indeed a global solution.

Example 21. $y' = x + y$ is a DE for which the variables cannot be separated.

No worries, very soon we will have several tools to solve this DE as well.

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ODEs vs PDEs

Important. Note that we are working with functions $y(x)$ of a single variable. This allows us to write simply y' for $\frac{\mathrm{d}}{\mathrm{d}x}y(x)$ without risk of confusion.

Of course, we may use different variables such as $x(t)$ and $x' = \frac{\mathrm{d}}{\mathrm{d}t}x(t)$, as long as this is clear from the context.

Differential equations that involve only derivatives with respect to a single variable are known as **ordinary differential equations** (ODEs).

On the other hand, differential equations that involve derivatives with respect to several variables are referred to as **partial differential equations** (PDEs).

Example 22. The DE

$$\left(\frac{\partial}{\partial x}\right)^2 u(x, y) + \left(\frac{\partial}{\partial y}\right)^2 u(x, y) = 0,$$

often abbreviated as $u_{xx} + u_{yy} = 0$, is a partial differential equation in two variables.

This particular PDE is known as **Laplace's equation** and describes, for ialsonstance, steady-state heat distributions.

https://en.wikipedia.org/wiki/Laplace%27s_equation

This and other fundamental PDEs will be discussed in Differential Equations II.

Existence and uniqueness of solutions

The following is a very general result that allows us to guarantee that "nice" IVPs must have a solution and that this solution is unique.

Comment. Note that any first-order DE can be written as $g(y', y, x) = 0$ where g is some function of three variables. Assuming that g is reasonable, we can solve for y' and rewrite such a DE as $y' = f(x, y)$ (for some, possibly complicated, function f).

Comment. To be precise, a solution to the IVP $y' = f(x, y)$, $y(a) = b$ is a function $y(x)$, defined on an interval I containing a, such that $y'(x) = f(x, y(x))$ for all $x \in I$ and $y(a) = b$.

Theorem 23. (existence and uniqueness) Consider the IVP $y' = f(x, y)$, $y(a) = b$.
If both $f(x, y)$ and $\frac{\partial}{\partial y}f(x, y)$ are continuous [in a rectangle] around (a, b), then the IVP has a unique solution in some interval $x \in (a - \delta, a + \delta)$ where $\delta > 0$.

Comment. The interval around a might be very small. In other words, the δ in the theorem could be very small.

Comment. Note that the theorem makes two important assertions. First, it says that there exists a **local solution**. Second, it says that this solution is unique. These two parts of the theorem are famous results usually attributed to Peano (existence) and Picard–Lindelöf (uniqueness).

Advanced comment. The condition about $\frac{\partial}{\partial y}f(x, y)$ is a bit technical (and not optimal). If we drop this condition, we still get existence but, in general, no longer uniqueness.

Advanced comment. The interval in which the solution is unique could be smaller than the interval in which it exists. In other words, it is possible that, away from the initial condition, the solution "forks" into two or more solutions. Note that this does not contradict the theorem because it only guarantees uniqueness on a small interval.

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Example 24. Consider the IVP $(x-y^2)y'=3x$, $y(4)=b$. For which choices of b does the existence and uniqueness theorem guarantee a unique (local) solution?

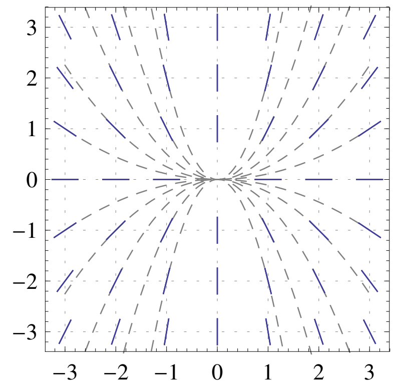
Solution. The IVP is $y'=f(x,y)$ with $f(x,y)=3x/(x-y^2)$. We compute that $\frac{\partial}{\partial y}f(x,y)=6xy/(x-y^2)^2$.

We observe that both $f(x,y)$ and $\frac{\partial}{\partial y}f(x,y)$ are continuous for all (x,y) with $x-y^2\neq 0$.

Note that $4-b^2\neq 0$ is equivalent to $b\neq\pm 2$.

Hence, if $b\neq\pm 2$, then the IVP locally has a unique solution by the existence and uniqueness theorem.

Example 25. Consider, again, the IVP $xy'=2y$, $y(a)=b$. Discuss existence and uniqueness of solutions (without solving).



Solution. The IVP is $y'=f(x,y)$ with $f(x,y)=2y/x$.

We compute that $\frac{\partial}{\partial y}f(x,y)=2/x$.

We observe that both $f(x,y)$ and $\frac{\partial}{\partial y}f(x,y)$ are continuous for all (x,y) with $x\neq 0$.

Hence, if $a\neq 0$, then the IVP locally has a unique solution by the existence and uniqueness theorem.

What happens in the case $a=0$?

Solution. In Example 18, we found that the DE $xy'=2y$ is solved by $y(x)=Cx^2$.

This means that the IVP with $y(0)=0$ has infinitely many solutions.

On the other hand, the IVP with $y(0)=b$ where $b\neq 0$ has no solutions. (This follows from the fact that there are no solutions to the DE besides $y(x)=Cx^2$. Can you see this by looking at the slope field?)

Example 26. Consider the IVP $y'=ky^2$, $y(a)=b$. Discuss existence and uniqueness of solutions.

Solution. The IVP is $y'=f(x,y)$ with $f(x,y)=ky^2$. We compute that $\frac{\partial}{\partial y}f(x,y)=2ky$.

We observe that both $f(x,y)$ and $\frac{\partial}{\partial y}f(x,y)$ are continuous for all (x,y).

Hence, for any initial conditions, the IVP locally has a unique solution by the existence and uniqueness theorem.

Example 27. Solve $y'=ky^2$.

Solution. Separate variables to get $\frac{1}{y^2}\frac{\mathrm{d}y}{\mathrm{d}x}=k$. Integrating $\int\frac{1}{y^2}\mathrm{d}y=\int k\,\mathrm{d}x$, we find $-\frac{1}{y}=kx+C$.

We solve for y to get $y=-\frac{1}{C+kx}=\frac{1}{D-kx}$ (with $D=-C$). That is the solution we verified earlier!

Comment. Note that we did not find the solution $y=0$ (it was "lost" when we divided by y^2). It is called a **singular solution** because it is not part of the **general solution** (the one-parameter family found above). However, note that we can obtain it from the general solution by letting $D\to\infty$.

Caution. We have to be careful about transforming our DE when using separation of variables: Just as the division by y^2 made us lose a solution, other transformations can add extra solutions which do not solve the original DE.

Here is a silly example (silly, because the transformation serves no purpose here) which still illustrates the point. The DE $(y-1)y'=(y-1)ky^2$ has the same solutions as $y'=ky^2$ plus the additional solution $y=1$ (which does not solve $y'=ky^2$).

Example 28. Solve the IVP $y'=y^2$, $y(0)=1$.

Solution. From the previous example with $k=1$, we know that $y(x)=\frac{1}{D-x}$.

Using $y(0)=1$, we find that $D=1$ so that the unique solution to the IVP is $y(x)=\frac{1}{1-x}$.

Comment. Note that we already concluded the uniqueness from the existence and uniqueness theorem.

On the other hand, note that $y(x)=\frac{1}{1-x}$ is only valid on $(-\infty,1)$ and that it cannot be continuously extended past $x=1$; it is only a local solution.

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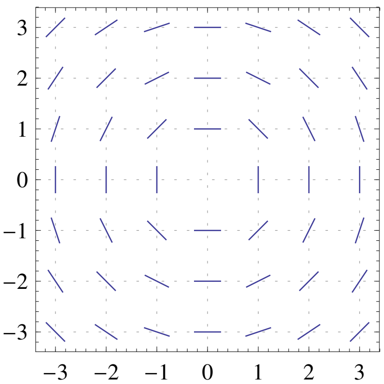
Notes for Lecture 6 Fri, 1/31/2025

Review. Existence and uniqueness theorem (Theorem 23) for an IVP $y' = f(x, y)$, $y(a) = b$:
If $f(x, y)$ and $\frac{\partial}{\partial y} f(x, y)$ are continuous around (a, b) then, locally, the IVP has a unique solution.

Example 29. Consider, again, the IVP $y' = -x / y$, $y(a) = b$. Discuss existence and uniqueness of solutions (without solving).



Solution. The IVP is $y' = f(x, y)$ with $f(x, y) = -x / y$.

We compute that $\frac{\partial}{\partial y} f(x, y) = x / y^2$.

We observe that both $f(x, y)$ and $\frac{\partial}{\partial y} f(x, y)$ are continuous for all (x, y) with $y \neq 0$.

Hence, if $b \neq 0$, then the IVP locally has a unique solution by the existence and uniqueness theorem.

Comment. In Example 14, we found that the DE $y' = -x / y$ is solved by $y(x) = \pm\sqrt{D - x^2}$.

Assume $b > 0$ (things work similarly for $b < 0$). Then $y(x) = \sqrt{D - x^2}$ solves the IVP (we need to choose D so that $y(a) = b$) if we choose $D = a^2 + b^2$. This confirms that there exists a solution. On the other hand, uniqueness means that there can be no other solution to the IVP than this one.

What happens in the case $b = 0$?

Solution. In this case, the existence and uniqueness theorem does not guarantee anything. If $a \neq 0$, then $y(x) = \sqrt{a^2 - x^2}$ and $y(x) = -\sqrt{a^2 - x^2}$ both solve the IVP (so we certainly don't have uniqueness), however only in a weak sense: namely, both of these solutions are not valid locally around $x = a$ but only in an interval of which a is an endpoint (for instance, the IVP $y' = -x / y$, $y(2) = 0$ is solved by $y(x) = \pm\sqrt{4 - x^2}$ but both of these solutions are only valid on the interval $[-2, 2]$ which ends at 2, and neither of these solutions can be extended past 2).

Example 30. Consider the initial value problem $(y^2 - 1)y' + \sin(x) = x^2$, $y(a) = b$. For which values of a and b can we guarantee existence and uniqueness of a (local) solution?

Solution. Let us write $y' = f(x, y)$ with $f(x, y) = \frac{x^2 - \sin(x)}{y^2 - 1}$. Then $\frac{\partial}{\partial y} f(x, y) = -\frac{x^2 - \sin(x)}{(y^2 - 1)^2} \cdot 2y$.

Both $f(x, y)$ and $\frac{\partial}{\partial y} f(x, y)$ are continuous for all (x, y) with $y \neq \pm 1$.

Hence, if $b \neq \pm 1$, then the IVP locally has a unique solution by the existence and uniqueness theorem.

Linear first-order DEs

A **linear differential equation** is one where the function y and its derivatives only show up linearly (i.e. there are no terms such as y^2, $1 / y$, $\sin(y)$ or $y \cdot y'$).

As such, the most general linear first-order DE is of the form

$$A(x)y' + B(x)y + C(x) = 0.$$

Such a DE can be rewritten in the following "**standard form**" by dividing by $A(x)$ and rearranging:

(linear first-order DE in standard form)

$$y' + P(x)y = Q(x)$$

We will use this standard form when solving linear first-order DEs.

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Example 31. (extra "warmup") Solve $\frac{\mathrm{d}y}{\mathrm{d}x} = 2xy^2$.

Solution. (separation of variables) $\frac{1}{y^2}\frac{\mathrm{d}y}{\mathrm{d}x} = 2x$, $-\frac{1}{y} = x^2 + C$.

Hence the general solution is $y = \frac{1}{D - x^2}$. [There also is the singular solution $y = 0$.]

Solution. (in other words) Note that $\frac{1}{y^2}\frac{\mathrm{d}y}{\mathrm{d}x} = 2x$ can be written as $\frac{\mathrm{d}}{\mathrm{d}x}\left[-\frac{1}{y}\right] = \frac{\mathrm{d}}{\mathrm{d}x}[x^2]$.
From there it follows that $-\frac{1}{y} = x^2 + C$, as above.

We now use the idea of writing both sides as a derivative (which we then integrate!) to also solve DEs that are not separable. We will be able to handle all first-order linear DEs this way.

The multiplication by $\frac{1}{y^2}$ will be replaced by multiplication with a so-called **integrating factor**.

Example 32. Solve $y' = x - y$.

Comment. Note that we cannot use separation of variables this time.

Solution. Rewrite the DE as $y' + y = x$.
Next, multiply both sides with e^x (we will see in a little bit how to find this "integrating factor") to get

$$\underbrace{e^x y' + e^x y}_{= \frac{\mathrm{d}}{\mathrm{d}x}[e^x y]} = xe^x.$$

The "magic" part is that we are able to realize the new left-hand side as a derivative!
We can then integrate both sides to get

$$e^x y = \int xe^x \mathrm{d}x = xe^x - e^x + C.$$

From here it follows that $y = x - 1 + Ce^{-x}$.
Comment. For the final integral, we used that $\int xe^x \mathrm{d}x = xe^x - \int e^x \mathrm{d}x = xe^x - e^x + C$ which follows, for instance, via integration by parts (with $f(x) = x$ and $g'(x) = e^x$ in the formula reviewed below).

Review. The multiplication rule $(fg)' = f'g + fg'$ implies $fg = \int f'g + \int fg'$.

The latter is equivalent to **integration by parts**:

$$\int f(x)g'(x)\mathrm{d}x = f(x)g(x) - \int f'(x)g(x)\mathrm{d}x$$

Comment. Sometimes, one writes $g'(x)\mathrm{d}x = \mathrm{d}g(x)$.

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In general, we can solve any **linear first-order DE** $y' + P(x)y = Q(x)$ in this way.

- We want to multiply with an **integrating factor** $f(x)$ such that the left-hand side of the DE becomes

$$f(x)y' + f(x)P(x)y = \frac{\mathrm{d}}{\mathrm{d}x}[f(x)y].$$

Since $\frac{\mathrm{d}}{\mathrm{d}x}[f(x)y] = f(x)y' + f'(x)y$, we need $f'(x) = f(x)P(x)$ for that.

- Check that $f(x) = \exp\left(\int P(x)\mathrm{d}x\right)$ has this property.

Comment. This follows directly from computing the derivative of this $f(x)$ via the chain rule.

Homework. On the other hand, note that finding f meant solving the DE $f' = P(x) f$. This is a separable DE. Solve it by separation of variables to arrive at the above formula for $f(x)$ yourself.

Just to make sure. There is no difference between $\exp(x)$ and e^x. Here, we prefer the former notation for typographical reasons.

With that integrating factor, we have the following recipe for solving any linear first-order equation:

(solving linear first-order DEs)

(a) Write the DE in the **standard form** $y' + P(x)y = Q(x)$.

(b) Compute the **integrating factor** as $f(x) = \exp\left(\int P(x)\mathrm{d}x\right)$.

[We can choose any constant of integration.]

(c) Multiply the DE from part (a) by $f(x)$ to get

$$\underbrace{f(x)y' + f(x)P(x)y}_{= \frac{\mathrm{d}}{\mathrm{d}x}[f(x)y]} = f(x)Q(x).$$

(d) Integrate both sides to get

$$f(x)y = \int f(x)Q(x)\mathrm{d}x + C.$$

Then solve for y by dividing by $f(x)$.

Comment. For better understanding, we prefer to go through the above steps. On the other hand, we can combine these steps into the following formula for the general solution of $y' + P(x)y = Q(x)$:

$$y = \frac{1}{f(x)}\left(\int f(x)Q(x)\mathrm{d}x + C\right) \quad \text{where } f(x) = e^{\int P(x)\mathrm{d}x}$$

Existence and uniqueness. Note that the solution we construct exists on any interval on which P and Q are continuous (not just on some possibly very small interval). This is better than what the existence and uniqueness theorem (Theorem 23) can guarantee. This is one of the many ways in which linear DEs have particularly nice properties compared to DEs in general.

Example 33. Solve $xy' = 2y + 1$, $y(-2) = 0$.

Solution. This is a linear first-order DE.

(a) Rewrite the DE as $\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x)$ (standard form) with $P(x) = -\frac{2}{x}$ and $Q(x) = \frac{1}{x}$.

(b) The integrating factor is $f(x) = \exp\left(\int P(x)\mathrm{d}x\right) = e^{-2\ln|x|} = e^{-2\ln(-x)} = (-x)^{-2} = \frac{1}{x^2}$.

Here, we used that, at least locally, $x < 0$ (because the initial condition is $x = -2 < 0$) so that $|x| = -x$.

(c) Multiply the DE (in standard form) by $f(x) = \frac{1}{x^2}$ to get

$$\underbrace{\frac{1}{x^2}\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{2}{x^3}y}_{= \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{x^2}y\right]} = \frac{1}{x^3}.$$

(d) Integrate both sides to get

$$\frac{1}{x^2}y = \int \frac{1}{x^3}\mathrm{d}x = -\frac{1}{2x^2} + C.$$

Hence, the general solution is $y(x) = -\frac{1}{2} + Cx^2$.

Solving $y(-2) = -\frac{1}{2} + 4C = 0$ for C yields $C = \frac{1}{8}$. Thus, the (unique) solution to the IVP is $y(x) = \frac{1}{8}x^2 - \frac{1}{2}$.

Example 34. (extra) Solve $y' = 2y + 3x - 1$, $y(0) = 2$.

Solution. This is a linear first-order DE.

(a) Rewrite the DE as $\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x)$ (standard form) with $P(x) = -2$ and $Q(x) = 3x - 1$.

(b) The integrating factor is $f(x) = \exp\left(\int P(x)\mathrm{d}x\right) = e^{-2x}$.

(c) Multiply the DE (in standard form) by $f(x) = e^{-2x}$ to get

$$\underbrace{e^{-2x}\frac{\mathrm{d}y}{\mathrm{d}x} - 2e^{-2x}y}_{= \frac{\mathrm{d}}{\mathrm{d}x}[e^{-2x}y]} = (3x - 1)e^{-2x}.$$

(d) Integrate both sides to get

$$\begin{aligned} e^{-2x}y &= \int (3x-1)e^{-2x}\mathrm{d}x \\ &= 3\int xe^{-2x}\mathrm{d}x - \int e^{-2x}\mathrm{d}x \\ &= 3\left(-\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x}\right) - \left(-\frac{1}{2}e^{-2x}\right) + C \\ &= -\frac{3}{2}xe^{-2x} - \frac{1}{4}e^{-2x} + C. \end{aligned}$$

Here, we used that $\int xe^{-2x}\mathrm{d}x = -\frac{1}{2}xe^{-2x} + \frac{1}{2}\int e^{-2x}\mathrm{d}x = -\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x}$ (for instance, via integration by parts with $f(x) = x$ and $g'(x) = e^{-2x}$).

Hence, the general solution is $y(x) = -\frac{3}{2}x - \frac{1}{4} + Ce^{2x}$.

Solving $y(0) = -\frac{1}{4} + C = 2$ for C yields $C = \frac{9}{4}$.

In conclusion, the (unique) solution to the IVP is $y(x) = -\frac{3}{2}x - \frac{1}{4} + \frac{9}{4}e^{2x}$.

Review. We can solve linear first-order DEs using **integrating factors**.

First, put the DE in standard form $y' + P(x)y = Q(x)$. Then $f(x) = \exp\left(\int P(x)\mathrm{d}x\right)$ is the integrating factor.

The key is that we get on the left-hand side $f(x)y' + f(x)P(x)y = \frac{\mathrm{d}}{\mathrm{d}x}[f(x)\,y]$. We can therefore integrate both sides with respect to x (the right-hand side is $f(x)Q(x)$ which is just a function depending on x—not y!).

Example 35. Solve $x^2\,y' = 1 - xy + 2x$, $y(1) = 3$.

Solution. This is a linear first-order DE. We can therefore solve it according to the recipe above.

(a) Rewrite the DE as $\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x)$ (standard form) with $P(x) = \frac{1}{x}$ and $Q(x) = \frac{1}{x^2} + \frac{2}{x}$.

(b) The integrating factor is $f(x) = \exp\left(\int P(x)\mathrm{d}x\right) = e^{\ln x} = x$.

Here, we could write $\ln x$ instead of $\ln|x|$ because the initial condition tells us that $x > 0$, at least locally.
Comment. We can also choose a different constant of integration but that would only complicate things.

(c) Multiply the DE (in standard form) by $f(x) = x$ to get

$$\underbrace{x\frac{\mathrm{d}y}{\mathrm{d}x} + y}_{= \frac{\mathrm{d}}{\mathrm{d}x}[xy]} = \frac{1}{x} + 2.$$

(d) Integrate both sides to get (again, we use that $x > 0$ to avoid having to use $|x|$)

$$xy = \int\left(\frac{1}{x} + 2\right)\mathrm{d}x = \ln x + 2x + C.$$

Using $y(1) = 3$ to find C, we get $1 \cdot 3 = \ln(1) + 2 \cdot 1 + C$ which results in $C = 3 - 2 = 1$.

Hence, the (unique) solution to the IVP is $y = \frac{\ln(x) + 2x + 1}{x}$.

Substitutions in DEs

Example 36. (review) Using substitution, compute $\int \frac{x}{1 + x^2}\mathrm{d}x$.

Solution. We substitute $u = 1 + x^2$. In that case, $\mathrm{d}u = 2x\,\mathrm{d}x$.
$\int \frac{x}{1 + x^2}\mathrm{d}x = \frac{1}{2}\int \frac{1}{u}\mathrm{d}u = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln(1 + x^2) + C$

Comment. Why were we allowed to drop the absolute value in the logarithm?

Review. On the other hand, recall that $\int \frac{1}{1 + x^2}\mathrm{d}x = \arctan(x) + C$.

Example 37. Solve $\frac{\mathrm{d}y}{\mathrm{d}x} = (x + y)^2$.

First things first. Is this DE separable? Is it linear? (No to both but make sure that this is clear to you.)
This means that our previous techniques are not sufficient to solve this DE.

Solution. Looking at the right-hand side, we have a feeling that the substitution $u = x + y$ might simplify things.

Then $y = u - x$ and, therefore, $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}x} - 1$.

Using these, the DE translates into $\frac{\mathrm{d}u}{\mathrm{d}x} - 1 = u^2$. This is a separable DE: $\frac{1}{1 + u^2}\mathrm{d}u = \mathrm{d}x$

After integration, we find $\arctan(u) = x + C$ and, thus, $u = \tan(x + C)$.

The solution of the original DE is $y = u - x = \tan(x + C) - x$.

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Useful substitutions

The previous example illustrates that different substitutions can help to solve a given DE.

Choosing the right substitution is difficult in general. The following is a compilation of important cases that are easy to spot and for which the listed substitutions are guaranteed to succeed:

- $y' = F\left(\frac{y}{x}\right)$

  Set $u = \frac{y}{x}$. Then $y = ux$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = x\frac{\mathrm{d}u}{\mathrm{d}x} + u$. We get $x\frac{\mathrm{d}u}{\mathrm{d}x} + u = F(u)$. This DE is always separable.

  Caution. The DE $y' = F\left(\frac{y}{x}\right)$ is sometimes called a "homogeneous equation". However, we will soon discuss homogeneous linear differential equations, where the label homogeneous means something different (though in both cases, there is a common underlying reason).

- $y' = F(ax + by)$

  Set $u = ax + by$. Then $y = \frac{1}{b}(u - ax)$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{b}\left(\frac{\mathrm{d}u}{\mathrm{d}x} - a\right)$.

  The new DE is $\frac{1}{b}\left(\frac{\mathrm{d}u}{\mathrm{d}x} - a\right) = F(u)$ or, simplified, $\frac{\mathrm{d}u}{\mathrm{d}x} = a + bF(u)$. This DE is always separable.

- $y' = F(x)y + G(x)y^n$ (This is called a **Bernoulli equation**.)

  Set $u = y^{1-n}$. The resulting DE is always linear.

  Details. If $u = y^{1-n}$ then $y = u^{1/(1-n)}$ and, thus, $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{1-n}u^{n/(1-n)}\frac{\mathrm{d}u}{\mathrm{d}x}$. $\left[\frac{1}{1-n} - 1 = \frac{n}{1-n}\right]$

  The new DE is $\frac{1}{1-n}u^{n/(1-n)}\frac{\mathrm{d}u}{\mathrm{d}x} = F(x)u^{1/(1-n)} + G(x)u^{n/(1-n)}$.

  Dividing both sides by $u^{n/(1-n)}$, the DE simplifies to $\frac{1}{1-n}\frac{\mathrm{d}u}{\mathrm{d}x} = F(x)u + G(x)$ which is a linear DE.

  Comment. The original DE has the trivial solution $y = 0$. Do you see where we lost that solution?

Example 38. Solve $\frac{\mathrm{d}y}{\mathrm{d}x} = (2x - 3y)^2 + \frac{2}{3}$, $y(1) = \frac{1}{3}$.

Solution. This is of the form $y' = F(2x - 3y)$ with $F(t) = t^2 + \frac{2}{3}$.

Therefore, as suggested by our list of useful substitutions, we substitute $u = 2x - 3y$.

Then $y = \frac{1}{3}(2x - u)$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{3}\left(2 - \frac{\mathrm{d}u}{\mathrm{d}x}\right)$.

The new DE is $\frac{1}{3}\left(2 - \frac{\mathrm{d}u}{\mathrm{d}x}\right) = u^2 + \frac{2}{3}$ or, simplified, $\frac{\mathrm{d}u}{\mathrm{d}x} = -3u^2$.

This DE is separable: $u^{-2}\mathrm{d}u = -3\mathrm{d}x$. After integration, $-\frac{1}{u} = -3x + C$.

We conclude that $u = \frac{1}{3x - C}$ and, hence, $y(x) = \frac{1}{3}(2x - u) = \frac{2}{3}x - \frac{1}{3}\frac{1}{3x - C}$.

Solving $y(1) = \frac{2}{3} - \frac{1}{3}\frac{1}{3 - C} = \frac{1}{3}$ for C leads to $C = 2$.

Hence, the unique solution of the IVP is $y(x) = \frac{2}{3}x - \frac{1}{3(3x - 2)}$.

Example 39. (homework) Consider the DE $x\frac{\mathrm{d}y}{\mathrm{d}x} = y + y^2 f(x)$.

(a) Substitute $u = \frac{y}{x}$. Is the resulting DE separable or linear?

(b) Substitute $v = \frac{1}{y}$. Is the resulting DE separable or linear?

(c) Solve each of the new DEs.

Solution.

(a) Set $u = \frac{y}{x}$. Then $y = ux$ and, thus, $\frac{\mathrm{d}y}{\mathrm{d}x} = x\frac{\mathrm{d}u}{\mathrm{d}x} + u$.

Using these, the DE translates into $x\left(x\frac{\mathrm{d}u}{\mathrm{d}x} + u\right) = ux + (ux)^2 f(x)$.

This DE simplifies to $\frac{\mathrm{d}u}{\mathrm{d}x} = u^2 f(x)$. This is a separable DE.

(b) Set $v = \frac{1}{y}$. Then $y = \frac{1}{v}$ and, thus, $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{v^2}\frac{\mathrm{d}v}{\mathrm{d}x}$.

Using these, the DE translates into $x\left(-\frac{1}{v^2}\frac{\mathrm{d}v}{\mathrm{d}x}\right) = \frac{1}{v} + \frac{1}{v^2} f(x)$.

This DE simplifies to $x\frac{\mathrm{d}v}{\mathrm{d}x} = -v - f(x)$. This is a linear DE.

(c) Let us write $F(x)$ for an antiderivative of $f(x)$.

- The DE $\frac{\mathrm{d}u}{\mathrm{d}x} = u^2 f(x)$ from the first part is separable: $u^2 \mathrm{d}u = f(x)\mathrm{d}x$.
  After integration, we find $-\frac{1}{u} = F(x) + C$.
  Since $u = \frac{y}{x}$, this becomes $-\frac{x}{y} = F(x) + C$.
  The general solution of the initial DE therefore is $y = -\frac{x}{F(x) + C}$.

- The DE $x\frac{\mathrm{d}v}{\mathrm{d}x} = -v - f(x)$ from the second part is linear. We apply our recipe:

  (a) Rewrite the DE as $\frac{\mathrm{d}v}{\mathrm{d}x} + P(x)v = Q(x)$ with $P(x) = 1/x$ and $Q(x) = -f(x)/x$.

  (b) The integrating factor is $\exp\left(\int P(x)\mathrm{d}x\right) = e^{\ln x} = x$.

  Comment. We should make a mental note that we assumed that $x > 0$. In the next step, however, we see that the integrating factor works for all x.

  (c) Multiply the (rewritten) DE by the integrating factor x to get $\underbrace{x\frac{\mathrm{d}v}{\mathrm{d}x} + v}_{= \frac{\mathrm{d}}{\mathrm{d}x}[xv]} = -f(x)$.

  (d) Integrate both sides to get $xv = -F(x) + C$.

  Since $v = \frac{1}{y}$, we find $\frac{x}{y} = -F(x) + C$.
  The general solution of the initial DE therefore is $y = -\frac{x}{F(x) - C}$.

Comment. Note that our two approaches led to the same general solution (from the existence and uniqueness theorem, we can see that this must be the case). One of the formulas features $+C$ while the other features $-C$. However, that makes no difference because C is a free parameter (we could have given them different names if we preferred).

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Example 40. Solve $(x-y)\frac{dy}{dx}=x+y$.

Solution. Divide the DE by x to get $\left(1-\frac{y}{x}\right)\frac{dy}{dx}=1+\frac{y}{x}$. This is a DE of the form $y'=F\left(\frac{y}{x}\right)$.

We therefore substitute $u=\frac{y}{x}$. Then $y=ux$ and $\frac{dy}{dx}=x\frac{du}{dx}+u$.

The resulting DE is $(x-ux)\left(x\frac{du}{dx}+u\right)=x+ux$, which simplifies to $x(1-u)\frac{du}{dx}=1+u^2$.

This DE is separable: $\frac{1-u}{1+u^2}du=\frac{1}{x}dx$

Integrating both sides, we find $\arctan(u)-\frac{1}{2}\ln(1+u^2)=\ln|x|+C$.

Setting $u=y/x$, we get the (general) implicit solution $\arctan(y/x)-\frac{1}{2}\ln(1+(y/x)^2)=\ln|x|+C$.

Comment. We used $\int\frac{1}{1+u^2}du=\arctan(u)+C$ and $\int\frac{x}{1+x^2}dx=\frac{1}{2}\ln(1+x^2)+C$ when integrating.

See Example 36 where we reviewed these integrals.

Example 41. Solve the IVP $\frac{dy}{dx}=2y-3xy^5$, $y(0)=1$.

Solution. This is an example of a Bernoulli equation (with $n=5$). We therefore substitute $u=y^{1-n}=y^{-4}$.

Accordingly, $y=u^{-1/4}$ and, thus, $\frac{dy}{dx}=-\frac{1}{4}u^{-5/4}\frac{du}{dx}$.

The new DE is $-\frac{1}{4}u^{-5/4}\frac{du}{dx}=2u^{-1/4}-3xu^{-5/4}$, which simplifies to $\frac{du}{dx}=-8u+12x$.

This is a linear first-order DE, which we solve according to our recipe:

(a) Rewrite the DE as $\frac{du}{dx}+P(x)u=Q(x)$ with $P(x)=8$ and $Q(x)=12x$.

(b) The integrating factor is $f(x)=\exp\left(\int P(x)dx\right)=e^{8x}$.

(c) Multiply the (rewritten) DE by $f(x)=e^{8x}$ to get

$$\underbrace{e^{8x}\frac{du}{dx}+8e^{8x}u}_{=\frac{d}{dx}[e^{8x}u]}=12xe^{8x}.$$

(d) Integrate both sides to get:

$$e^{8x}u=12\int xe^{8x}dx=12\left(\frac{1}{8}xe^{8x}-\frac{1}{8^2}e^{8x}\right)+C=\frac{3}{2}xe^{8x}-\frac{3}{16}e^{8x}+C$$

Here we used that $\int xe^{ax}dx=\frac{1}{a}xe^{ax}-\frac{1}{a^2}e^{ax}$. (Integration by parts!)

The general solution of the DE for u therefore is $u=\frac{3}{2}x-\frac{3}{16}+Ce^{-8x}$.

Correspondingly, the general solution of the initial DE is $y=u^{-1/4}=1/\sqrt[4]{\frac{3}{2}x-\frac{3}{16}+Ce^{-8x}}$.

Using $y(0)=1$, we find $1=1/\sqrt[4]{C-\frac{3}{16}}$ from which we obtain $C=1+\frac{3}{16}=\frac{19}{16}$.

The unique solution to the IVP therefore is $y=1/\sqrt[4]{\frac{3}{2}x-\frac{3}{16}+\frac{19}{16}e^{-8x}}$.

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Solving simple 2nd order DEs

We have the following two useful substitutions for certain simple DEs of order 2:

- $F(y'', y', x) = 0$ (2nd order with "y missing")

  Set $u = y' = \frac{dy}{dx}$. Then $y'' = \frac{du}{dx}$. We get the first-order DE $F\left(\frac{du}{dx}, u, x\right) = 0$.

- $F(y'', y', y) = 0$ (2nd order with "x missing")

  Set $u = y' = \frac{dy}{dx}$. Then $y'' = \frac{du}{dx} = \frac{du}{dy} \cdot \frac{dy}{dx} = \frac{du}{dy} \cdot u$. We get the first-order DE $F\left(u\frac{du}{dy}, u, y\right) = 0$.

Example 42. Solve $y'' = x - y'$.

Solution. We substitute $u = y'$, which results in the first-order DE $u' = x - u$.
This DE is linear and, using our recipe (see below for the details), we can solve it to find $u = x - 1 + Ce^{-x}$.
Since $y' = u$, we conclude that the general solution is $y = \int (x - 1 + Ce^{-x})dx = \frac{1}{2}x^2 - x - Ce^{-x} + D$.

Important comment. This is a DE of order 2. Hence, as expected, the general solution has two free parameter.

Solving the linear DE. To solve $u' = x - u$ (also see Example 32, where we had solved this DE before), we

(a) rewrite the DE as $\frac{du}{dx} + P(x)u = Q(x)$ with $P(x) = 1$ and $Q(x) = x$.

(b) The integrating factor is $f(x) = \exp\left(\int P(x)dx\right) = e^x$.

(c) Multiply the (rewritten) DE by $f(x) = e^x$ to get $\underbrace{e^x\frac{du}{dx} + e^x u}_{= \frac{d}{dx}[e^x u]} = xe^x$.

(d) Integrate both sides to get (using integration by parts): $e^x u = \int xe^x dx = xe^x - e^x + C$

Hence, the general solution of the DE for u is $u = x - 1 + Ce^{-x}$, which is what we used above.

Example 43. (homework) Solve the IVP $y'' = x - y'$, $y(0) = 1$, $y'(0) = 2$.

Solution. As in the previous example, we find that the general solution to the DE is $y(x) = \frac{1}{2}x^2 - x - Ce^{-x} + D$.
Using $y'(x) = x - 1 + Ce^{-x}$ and $y'(0) = 2$, we find that $2 = -1 + C$. Hence, $C = 3$.
Then, using $y(x) = \frac{1}{2}x^2 - x - 3e^{-x} + D$ and $y(0) = 1$, we find $1 = -3 + D$. Hence, $D = 4$.
In conclusion, the unique solution to the IVP is $y(x) = \frac{1}{2}x^2 - x - 3e^{-x} + 4$.

Example 44. (extra) Find the general solution to $y'' = 2yy'$.

Solution. We substitute $u = y' = \frac{dy}{dx}$. Then $y'' = \frac{du}{dx} = \frac{du}{dy} \cdot \frac{dy}{dx} = \frac{du}{dy} \cdot u$.
Therefore, our DE turns into $u\frac{du}{dy} = 2yu$.
Dividing by u, we get $\frac{du}{dy} = 2y$. [Note that we lose the solution $u = 0$, which gives the singular solution $y = C$.]
Hence, $u = y^2 + C$. It remains to solve $y' = y^2 + C$. This is a separable DE.
$\frac{1}{C + y^2}dy = dx$. Let us restrict to $C = D^2 \geqslant 0$ here. (This means we will only find "half" of the solutions.)
$\int \frac{1}{D^2 + y^2}dy = \frac{1}{D^2}\int \frac{1}{1 + (y/D)^2}dy = \frac{1}{D}\arctan(y/D) = x + A$.
Solving for y, we find $y = D\tan(Dx + AD) = D\tan(Dx + B)$. $[B = AD]$

Applications of DEs & Modeling

The exponential model of population growth

If $P(t)$ is the size of a population (eg. of bacteria) at time t, then the rate of change $\frac{\mathrm{d}P}{\mathrm{d}t}$ might, from biological considerations, be (nearly) proportional to $P(t)$.

More down to earth, this is just saying "for a population 5 times as large, we expect 5 times as many babies".

Say, we have a population of $P = 100$ and $P' = 3$, meaning that the population changes by 3 individuals per unit of time. By how much do we expect a population of $P = 500$ to change? (Think about it for a moment!)

Without further information, we would probably expect the population of $P = 500$ to change by $5 \cdot 3 = 15$ individuals per unit of time, so that $P' = 15$ in that case. This is what it means for P' to be proportional to P. In formulas, it means that P'/P is constant or, equivalently, that $P' = kP$ for a proportionality constant k.

Comment. "Population" might sound more specific than it is. It could also refer to rather different populations such as amounts of money (finance) or amounts of radioactive material (physics).

For instance, thinking about an amount $P(t)$ of money in a bank account at time t, we would also expect $\frac{\mathrm{d}P}{\mathrm{d}t}$ (the money per time that we gain from receiving interest) to be proportional to $P(t)$.

The corresponding **mathematical model** is described by the DE $\frac{\mathrm{d}P}{\mathrm{d}t} = kP$ where k is the constant of proportionality.

Example 45. Determine all solutions to the DE $\frac{\mathrm{d}P}{\mathrm{d}t} = kP$.

Solution. We easily guess and then verify that $P(t) = Ce^{kt}$ is a solution. (Alternatively, we can find this solution via separation of variables or because this is a linear DE. Do it both ways!)

Moreover, it follows from the existence and uniqueness theorem that there cannot be further solutions. (Alternatively, we can conclude this from our solving process (separation of variables or our approach to linear DEs only lose solutions when we divide by zero and we can consider those cases separately)).

Mathematics therefore tells us that the (only) solutions to this DE are given by $P(t) = Ce^{kt}$ where C is some constant.

Hence, populations satisfying the assumption from biology necessarily exhibit exponential growth.

The **exponential model** with growth rate k is

$$\frac{\mathrm{d}P}{\mathrm{d}t} = kP.$$

The general solution is $P(t) = Ce^{kt}$ where $C = P(0)$.

Example 46. Let $P(t)$ describe the size of a population at time t. Suppose $P(0) = 100$ and $P(1) = 300$. Under the exponential model of population growth, find $P(t)$.

Solution. $P(t)$ solves the DE $\frac{\mathrm{d}P}{\mathrm{d}t} = kP$ and therefore is of the form $P(t) = Ce^{kt}$.

We now use the two data points to determine both C and k.

$Ce^{k \cdot 0} = C = 100$ and $Ce^{k} = 100e^{k} = 300$. Hence $k = \ln(3)$ and $P(t) = 100e^{\ln(3)t} = 100 \cdot 3^{t}$.

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The logistic model of population growth

If the population is constrained by resources, then $\frac{\mathrm{d}P}{\mathrm{d}t} = kP$ is not a good model. A model to take that into account is $\frac{\mathrm{d}P}{\mathrm{d}t} = kP\left(1 - \frac{P}{M}\right)$. This is the **logistic equation**.

M is called the carrying capacity:

- Note that if $P \ll M$ then $1 - \frac{P}{M} \approx 1$ and we are back to the simpler exponential model. This means that the population P will grow (nearly) exponentially if P is much less than the carrying capacity M.
- On the other hand, if $P > M$ then $1 - \frac{P}{M} < 0$ so that (assuming $k > 0$) $\frac{\mathrm{d}P}{\mathrm{d}t} < 0$, which means that the population P is shrinking if it exceeds the carrying capacity M.

Comment. If $P(t)$ is the size of a population, then P'/P can be interpreted as its *per capita growth rate*.
Note that in the exponential model we have that $P'/P = k$ is constant.
On the other hand, in the logistic model we have that $P'/P = k(1 - P/M)$ is a linear function.

The **logistic model** with growth rate k and carrying capacity M is

$$\frac{\mathrm{d}P}{\mathrm{d}t} = kP\left(1 - \frac{P}{M}\right).$$

The general solution is $P(t) = \frac{M}{1 + Ce^{-kt}}$ where $C = \frac{M}{P(0)} - 1$.

Important. We will solve the logistic equation in detail in Example 49 to find the stated formula for $P(t)$. At this point, can you already see what technique we will be able to use? (We actually have two options!) Note that, even if we couldn't solve the DE, we can always verify that the stated $P(t)$ solves the DE by plugging in.

Example 47. Let $P(t)$ describe the size of a population at time t. Under the logistic model of population growth, what is $\lim_{t \to \infty} P(t)$?

Solution.

- If $k > 0$, then $e^{-kt} \to 0$ and it follows from $P(t) = \frac{M}{1 + Ce^{-kt}}$ that $\lim_{t \to \infty} P(t) = M$.
  In other words, the population will approach the carrying capacity in the long run.
- If $k = 0$, then we simply have $P(t) = \frac{M}{1 + C}$. In other words, the population remains constant.
  This is a corner case because the DE becomes $\frac{\mathrm{d}P}{\mathrm{d}t} = 0$.
- If $k < 0$, then $e^{-kt} \to \infty$ and it follows that $\lim_{t \to \infty} P(t) = 0$.
  In other words, the population will approach extinction in the long run.

Comment. There is also the trivial corner case arising from $P(0) = 0$ (then our C would be infinite), in which case $P(t) = 0$. We will always assume that we are not talking about a zero (or negative) population.

Example 48. (homework) A rising population is modeled by the equation $\frac{\mathrm{d}P}{\mathrm{d}t} = 400P - 2P^2$.

(a) When the population size stabilizes in the long term, how large will it be?

(b) Under which condition would the population size shrink?

(c) What is the population size when it is growing the fastest?

(d) If $P(0) = 10$, what is $P(t)$?

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Solution.

(a) Once the population reaches a stable level in the long term, we have $\frac{\mathrm{d}P}{\mathrm{d}t} = 0$ (no change in population size). Hence, $0 = 400P - 2P^2 = 2P(200 - P)$ which implies that $P = 0$ or $P = 200$. Since the population is rising, it will approach 200 in the long term.

Alternatively. Our DE matches the logistic equation $\frac{\mathrm{d}P}{\mathrm{d}t} = kP\left(1 - \frac{P}{M}\right)$ with $k = 400$ and $M = 200$.

(b) The population size would shrink if $\frac{\mathrm{d}P}{\mathrm{d}t} < 0$.

The DE tells us that is the case if and only if $400P - 2P^2 < 0$ or, equivalently, if $P > \frac{400}{2} = 200$.

Comment. In the logistic model, the population shrinks if it exceeds the carrying capacity.

(c) This is asking when $\frac{\mathrm{d}P}{\mathrm{d}t}$ (the population growth) is maximal.

The DE is telling us that this growth is $f(P) = 400P - 2P^2$. This a parabola that opens to the bottom. From Calculus, we know that it has a global maximum when $f'(P) = 0$.

$f'(P) = 400 - 4P = 0$ leads to $P = 100$.

Thus, the population is growing the fastest when its size is 100.

Comment. In the logistic model, the population is growing fastest when it is half the carrying capacity.

(d) We know that the general solution of the logistic equation is $P(t) = \frac{M}{1 + Ce^{-kt}} = \frac{200}{1 + Ce^{-400t}}$.

Using $P(0) = 10$, we find that $C = \frac{200}{10} - 1 = 19$.

Thus $P(t) = \frac{200}{1 + 19e^{-400t}}$.

Example 49. Solve the logistic equation $\frac{\mathrm{d}P}{\mathrm{d}t} = kP\left(1 - \frac{P}{M}\right)$.

Solution. This is a separable DE: $\frac{1}{P(1 - \frac{P}{M})}\mathrm{d}P = k\mathrm{d}t$.

To integrate the left-hand side, we use partial fractions: $\frac{1}{P(1 - \frac{P}{M})} = \frac{1}{P} + \frac{1/M}{1 - \frac{P}{M}} = \frac{1}{P} - \frac{1}{P - M}$.

After integrating, we obtain $\ln|P| - \ln|P - M| = kt + A$.

Equivalently, $\ln\left|\frac{P}{P - M}\right| = kt + A$ so that $\frac{P}{P - M} = \pm e^{kt + A} = Be^{kt}$ where $B = \pm e^A$.

Solving for P, we conclude that the general solution is

$$P(t) = \frac{BMe^{kt}}{Be^{kt} - 1} = \frac{M}{1 + Ce^{-kt}},$$

where we replaced the free parameter B with $C = -1/B$.

Initial population. Note that the initial population is $P(0) = \frac{M}{1 + C}$. Equivalently, $C = \frac{M}{P(0)} - 1$ which expresses the free parameter C in terms of the initial population.

Comment. Note that $B = \pm e^A$ can be any real number except 0. However, we can easily check that $B = 0$ also gives us a solution to the DE (namely, the trivial solution $P = 0$). This solution was "lost" when we divided by P to separate variables.

Exercise. Note that the logistic equation is also a Bernoulli equation. As an alternative to separation of variables, we can therefore solve it by transforming it to a linear DE via substitution.

Review of partial fractions. Recall that partial fractions tells us that fractions like $\frac{p(x)}{(x - r_1)(x - r_2)\cdots}$ (with the numerator of smaller degree than the denominator; and with the r_j distinct) can be written as a sum of terms of the form $\frac{A_j}{x - r_j}$ for suitable constants A_j.

In our case, this tells us that $\frac{1}{P(1 - P/M)} = \frac{A}{P} + \frac{B}{1 - P/M}$ for certain constants A and B.

Multiply both sides by P and set $P = 0$ to find $A = 1$.

Multiply both sides by $1 - P/M$ and set $P = M$ to find $B = 1/M$. This is what we used above.

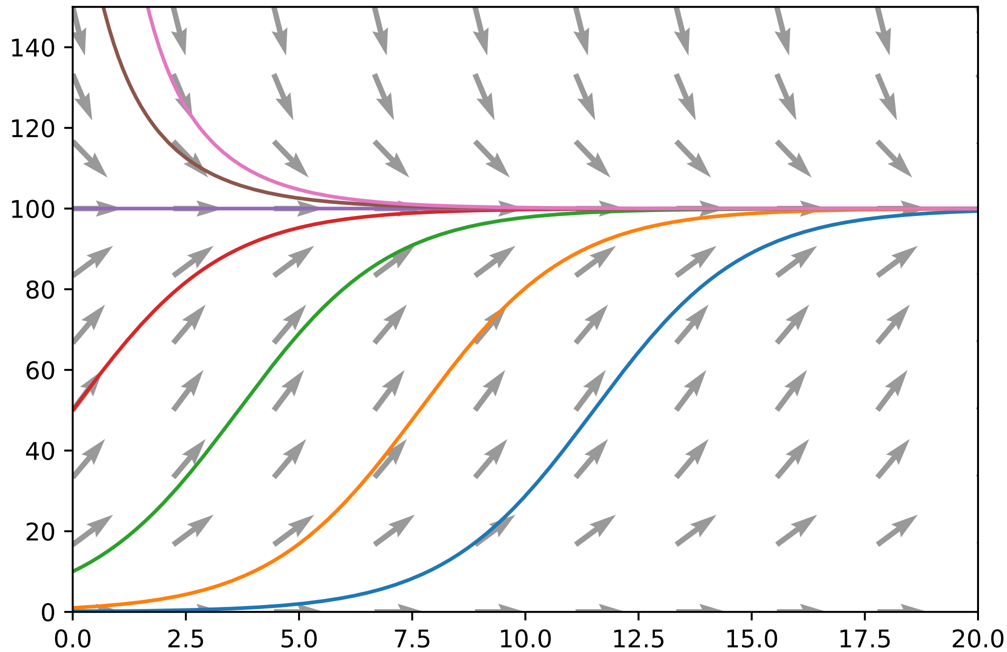
Review. The **logistic equation** is $\frac{\mathrm{d}P}{\mathrm{d}t} = kP\left(1 - \frac{P}{M}\right)$.

Here, k is the growth rate and M is the carrying capacity.

The general solution of the logistic equation is $P(t) = \frac{M}{1 + Ce^{-kt}}$ where $C = \frac{M}{P(0)} - 1$.

Example 50. Visualize the logistic equation for $k = 0.6$ and $M = 100$ using a slope field as well as by plotting some solution functions.

Solution.



In this slope field, we plotted the solutions $P(t)$ with $P(0) = 0.1, 1, 10, 50, 100, 200, 1000$.

Main challenge of modeling: a model has to be detailed enough to resemble the real world, yet simple enough to allow for mathematical analysis.

Extending the exponential model. Observe that the exponential model of population growth can be written as

$$\frac{P'}{P} = \text{constant}.$$

Thinking purely mathematically (generally not a good idea for modeling!), to extend the model, it might be sensible to replace constant (which we called k above) by the next simplest kind of function, namely a linear function in P. The resulting DE is the **logistic equation**.

Comment. Can you put into words why we replace constant by a function of P rather than a function of t? When would it be appropriate to add a dependence on t?

[A dependence on t would make sense if the "environment" changes over time. Without such a change, we expect that a population (say, of bacteria in our lab) behaves this week just as it would next week. The "law" behind its growth should not depend on t. The resulting differential equations are called **autonomous**.]

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Example 51. In a city with a fixed population N, the time rate of change of the number P of people who have heard a certain rumor is proportional to the product of P and $N-P$. Suppose initially 10% have heard the rumor and after a week this number has grown to 20%. What percentage will this number reach after one more week?

Solution. We are told that $\frac{\mathrm{d}P}{\mathrm{d}t}=\gamma P(N-P)$ as well as $P(0)=0.1N$ and $P(1)=0.2N$. We need $P(2)$.

Note that this is a logistic equation $\frac{\mathrm{d}P}{\mathrm{d}t}=kP\left(1-\frac{P}{N}\right)$ with $k=\gamma N$ and carrying capacity N.

It therefore has the general solution $P(t)=\dfrac{N}{1+Ce^{-kt}}$.

Using $P(0)=\frac{N}{1+C}=0.1N$, we find that $C=9$.

Using $P(1)=\frac{N}{1+9e^{-k}}=0.2N$, we further find that $e^{-k}=\frac{4}{9}$.

We could solve for k but note that it is more pleasing to use $e^{-kt}=(e^{-k})^t=\left(\frac{4}{9}\right)^t$ in our formula for $P(t)$.

We conclude that $P(t)=\dfrac{N}{1+9\left(\frac{4}{9}\right)^t}$.

In particular, $P(2)=\frac{N}{1+9\cdot\frac{16}{81}}=\frac{9}{25}N$ which is 36%.

Example 52. A scientist is claiming that a certain population $P(t)$ follows the logistic model of population growth. How many data points do you need to begin to verify that claim?

Solution. The general solution $P(t)=\dfrac{M}{1+Ce^{-kt}}$ to the logistic equation has 3 parameters.

Hence, we need 3 data points just to solve for their values.

Once we have 4 or more data points, we are able to test whether $P(t)$ conforms to the logistic model.

Important comment. Complicated models tend to have more degrees of freedom, which makes it easier to fit them to real world data (even if the model is not actually particularly appropriate). We therefore need to be cognizant about how much evidence is needed to decide that a given model is appropriate for the data.

Further population models

Let $P(t)$ be the size of the population that we wish to model at time t.

Denote with $\beta(t)$ and $\delta(t)$ the birth and death rate at time t, measured in number of births or deaths per unit of population per unit of time.

In the time interval $[t, t + \Delta t]$, we have that

$$\Delta P \approx \beta(t)P(t)\Delta t - \delta(t)P(t)\Delta t.$$

Comment. The reason that this is not an exact equation is that the rates $\beta(t)$ and $\delta(t)$ are allowed to change with t. In the above, we used these rates at time t for all times in $[t, t+\Delta t]$. This is a good approximation if Δt is small.

Divide both sides by Δt and let $\Delta t \to 0$ to obtain the general differential equation

$$\frac{\mathrm{d}P}{\mathrm{d}t} = (\beta(t) - \delta(t))P.$$

Given certain scenarios, we now make corresponding reasonable choices for $\beta(t)$ and $\delta(t)$.

- **(basic)** If the rates $\beta(t)$ and $\delta(t)$ are constant over time, the DE is $\frac{\mathrm{d}P}{\mathrm{d}t} = (\beta - \delta)P$.
  This is the exponential model of population growth.

- **(limited supply)** If supply is limited, the birth rate will decrease as P increases. The simplest such relationship would be a linear dependence, which would take the form $\beta(t) = \beta_0 - \beta_1 P$.

  On the other hand, we still assume that $\delta(t)$ is constant. (However, depending on circumstances, it could also be reasonable to assume that $\delta(t)$ increases as P increases.)

  With these assumptions, the corresponding DE is $\frac{\mathrm{d}P}{\mathrm{d}t} = (\beta_0 - \beta_1 P - \delta)P$.

  This is the logistic equation $\frac{\mathrm{d}P}{\mathrm{d}t} = kP(1 - P/M)$ with $k = \beta_0 - \delta$ and $\frac{k}{M} = \beta_1$.

- **(rare isolated species)** If the population consists of rare and isolated specimen which rely on chance encounters to reproduce, then it is reasonable to assume that the birth rate $\beta(t)$ is proportional to $P(t)$ (larger $P(t)$ means more possibilities for chance encounters). Once more, we assume that $\delta(t)$ constant.

  With these assumptions, the corresponding DE is $\frac{\mathrm{d}P}{\mathrm{d}t} = (kP - \delta)P$.
  This is, again, the logistic equation.

- **(rare isolated species with very long life)** As before, for a rare isolated population, it is reasonable to assume that $\beta(t)$ is proportional to $P(t)$. If, in addition, our specimen have very long life, then we would assume that $\delta(t) = 0$.

  The corresponding DE is $\frac{\mathrm{d}P}{\mathrm{d}t} = kP^2$. Solutions are $P(t) = \frac{1}{C - kt}$ where $P(0) = 1/C$. (Do it!)

  Comment. Note that $P(t) \to \infty$ as $t \to C/k$. This explosion (which implies population growth beyond exponential growth) emphasizes that we can only use the DE while our initial assumptions are satisfied. Here, the DE is no longer appropriate when our species is no longer rare because $P(t)$ is too large.

- **(spread of contagious incurable virus)** Let $P(t)$ count the number of infected population units among a (constant) total of N. Since the virus is incurable, we have $\delta(t) = 0$. On the other hand, it is reasonable to assume that $\beta(t)$ is proportional to $N - P$ (the number of people that can still be infected).

  The resulting DE is $\frac{\mathrm{d}P}{\mathrm{d}t} = kP(N - P)$. Once again, this is the logistic equation.

- **(harvesting)** Suppose that h population units are harvested each unit of time.

  Then the DE becomes $\frac{\mathrm{d}P}{\mathrm{d}t} = (\beta(t) - \delta(t))P - h$.

  For instance. $\frac{\mathrm{d}P}{\mathrm{d}t} = kP - h$ has the solution $P(t) = Ce^{kt} + h/k$. In that case, we get exponential growth if $C > 0$. Note that $P(0) = C + h/k$. In terms of the initial population $P(0)$, we therefore get exponential growth if $P(0) > h/k$. (Also see next example!)

Example 53. A biotech company is growing certain microorganisms in the lab. From experience they know that the growth (number of organisms per day) of the microorganisms is well modeled by an exponential model with proportionality constant $k=5$ (per day). What is the optimal rate (in number of organisms per day) at which the company can continually harvest the microorganisms?

Solution. (long version via solving the DE) Without harvesting, the growth is modeled by $\frac{\mathrm{d}P}{\mathrm{d}t}=5P$ (the exponential model). Here, P is the number of organisms and t measures time in days. (Always think about your units in applications!)

If harvesting occurs at the rate of h number of organisms per day, the population model needs to be adjusted to

$$\frac{\mathrm{d}P}{\mathrm{d}t}=5P-h.$$

Since h is a constant, we can solve this DE using separation of variables. Alternatively, the DE is linear and we can therefore solve it using an integrating factor. For practice, we do both:

- **(separation of variables)** Integrating $\frac{1}{5P-h}\mathrm{d}P=\mathrm{d}t$, we find $\frac{1}{5}\ln|5P-h|=t+C$, which we simplify to $|5P-h|=e^{5t+5C}$. It follows that $5P-h=\pm e^{5t}e^{5C}=Be^{5t}$ where we wrote $B=\pm e^{5C}$ (note that the sign is fixed and cannot change).
  Thus, the general solution of the DE is $P(t)=\frac{h}{5}+Ae^{5t}$ (where we wrote $A=\frac{B}{5}$).
- **(integrating factor)** Since this is a linear DE, we can solve it as follows:
  - We write the DE in the form $\frac{\mathrm{d}P}{\mathrm{d}t}-5P=-h$.
  - The integrating factor is $f(t)=\exp(\int -5\,\mathrm{d}t)=e^{-5t}$.
  - Multiply the (rewritten) DE by $f(t)$ to get $\underbrace{e^{-5t}\frac{\mathrm{d}P}{\mathrm{d}t}-5e^{-5t}P}_{=\frac{\mathrm{d}}{\mathrm{d}t}[e^{-5t}P]}=-he^{-5t}$.
  - Integrate both sides to get $e^{-5t}P=-h\int e^{-5t}\mathrm{d}t=\frac{h}{5}e^{-5t}+C$.

  Hence the general solution to the DE is $P(t)=\frac{h}{5}+Ce^{5t}$.

In either case, we found that $P(t)=\frac{h}{5}+Ce^{5t}$. In order to be able to continually harvest, we need to make sure that $C\geqslant 0$. In terms of the initial population, we get $P(0)=\frac{h}{5}+C$ so that $C=P(0)-\frac{h}{5}$.

Thus the condition $C\geqslant 0$ becomes $P(0)-\frac{h}{5}\geqslant 0$ or, equivalently, $h\leqslant 5P(0)$. Thus, the optimal rate of harvesting is $h=5P(0)$.

Solution. (short version) As before, we observe that, if harvesting occurs at the rate of h number of organisms per day, then our population model is

$$\frac{\mathrm{d}P}{\mathrm{d}t}=5P-h.$$

The crucial observation is that the optimal harvesting rate should occur if $\frac{\mathrm{d}P}{\mathrm{d}t}=0$ (if $\frac{\mathrm{d}P}{\mathrm{d}t}>0$, then the population grows indicating that we could have harvested at a higher rate; if $\frac{\mathrm{d}P}{\mathrm{d}t}<0$ then the population shrinks and, all else being equal, we should no longer be able to harvest at the optimal rate).

We thus get the condition $5P-h=0$. Since the population is constant (because of $\frac{\mathrm{d}P}{\mathrm{d}t}=0$) this is equivalent to $5P(0)-h=0$. Again, we conclude that the optimal rate of harvesting is $h=5P(0)$.

Application: Mixing problems

Example 54. A tank contains 20gal of pure water. It is filled with brine (containing 5lb/gal salt) at a rate of 3gal/min. At the same time, well-mixed solution flows out at a rate of 2gal/min. How much salt is in the tank after t minutes?

Solution.

(Part I. determining a DE) Let $x(t)$ denote the amount of salt (in lb) in the tank after time t (in min).

At time t, the concentration of salt (in lb/gal) in the tank is $\frac{x(t)}{V(t)}$ where $V(t) = 20 + (3-2)t = 20 + t$ is the volume (in gal) in the tank.

In the time interval $[t, t + \Delta t]$: $\Delta x \approx 3 \cdot 5 \cdot \Delta t - 2 \cdot \frac{x(t)}{V(t)} \cdot \Delta t$.

Hence, $x(t)$ solves the IVP $\frac{\mathrm{d}x}{\mathrm{d}t} = 15 - 2 \cdot \frac{x}{20+t}$ with $x(0) = 0$.

Comment. Can you explain why the equation for Δx is only approximate but why the final DE is exact? [Hint: $x(t)/V(t)$ is the concentration at time t but we are using it for Δx at other times as well.]

(Part II. solving the DE) Since this is a linear DE, we can solve it as follows:

- Write the DE in the standard form as $\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{2}{20+t} x = 15$.
- The integrating factor is $f(t) = \exp\left(\int \frac{2}{20+t} \mathrm{d}t\right) = \exp(2\ln|20+t|) = (20+t)^2$.
- Multiply the DE (in standard form) by $f(t) = (20+t)^2$ to get $\underbrace{(20+t)^2 \frac{\mathrm{d}x}{\mathrm{d}t} + 2(20+t)x}_{= \frac{\mathrm{d}}{\mathrm{d}t}[(20+t)^2 x]} = 15(20+t)^2$.
- Integrate both sides to get $(20+t)^2 x = 15 \int (20+t)^2 \mathrm{d}t = 5(20+t)^3 + C$.

Hence the general solution to the DE is $x(t) = 5(20+t) + \frac{C}{(20+t)^2}$. Using $x(0) = 0$, we find $C = -5 \cdot 20^3$.

We conclude that, after t minutes, the tank contains $x(t) = 5(20+t) - \frac{5 \cdot 20^3}{(20+t)^2}$ pounds of salt.

Comment. As a consequence, $x(t) \approx 5(20+t) = 5V(t)$ for large t. Can you explain why that makes perfect sense and why we could have predicted this from the very beginning (without deriving a DE and solving it)?

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Numerically "solving" DEs: Euler's method

Recall that the general form of a first-order initial value problem is

$$y' = f(x, y), \quad y(x_0) = y_0.$$

Further recall that, under mild assumptions on $f(x, y)$, such an IVP has a unique solution $y(x)$. We have learned some techniques for (exactly) solving DEs. On the other hand, many DEs that arise in practice cannot be solved by these techniques (or more fancy ones).

Instead, it is common in practice to approximate the solution $y(x)$ to our IVP. Euler's method is the simplest example of how this can done. The key idea is to locally approximate $y(x)$ by tangent lines:

Example 55. Suppose y solves the IVP $y' = f(x, y)$ with $y(x_0) = y_0$. Using the tangent line at (x_0, y_0), find an approximation for $y(x_1)$ where $x_1 = x_0 + h$.

The idea is that we choose the **step size** h to be sufficiently small so that the approximation is good enough.

Solution. The tangent line at (x_0, y_0) has slope $y'(x_0) = f(x_0, y_0)$ which is a number we can compute without solving the DE for $y(x)$. Hence, the equation for the tangent line is $T(x) = y_0 + f(x_0, y_0)(x - x_0)$.

We now use this tangent line as an approximation for the solution of the DE to find

$$y(x_1) \approx T(x_1) = y_0 + f(x_0, y_0)(x_1 - x_0) = y_0 + f(x_0, y_0)h.$$

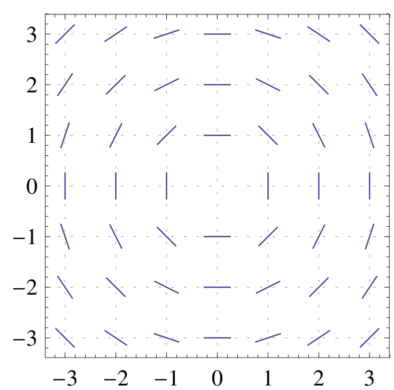
At this point, we have gone from our initial point (x_0, y_0) to a next (approximate) point (x_1, y_1). We now repeat what we did to get a third point (x_2, y_2) with $x_2 = x_1 + h$. Continuing in this way, we obtain Euler's method:

(Euler's method) To approximate the solution $y(x)$ of the IVP $y' = f(x, y)$, $y(x_0) = y_0$, we start with the point (x_0, y_0) and a step size h. We then compute:

$$\begin{aligned} x_{n+1} &= x_n + h \\ y_{n+1} &= y_n + h f(x_n, y_n) \end{aligned}$$

Example 56. Consider, again, the DE $y' = -x/y$.

We earlier produced the slope field on the right. We also used separation of variables to find that the solutions are circles $y(x) = \pm\sqrt{r^2 - x^2}$.



We know that the unique solution to the IVP with $y(0) = 2$ is $y(x) = \sqrt{4 - x^2}$. On the other hand, approximate the solution using Euler's method with $h = 1$ and 2 steps.

Solution. First, use just the slope field to sketch the 2 approximate points.

On the other hand, applying Euler's method with $f(x, y) = -x/y$:

$$\begin{array}{ll} x_0 = 0 & y_0 = 2 \\ x_1 = 1 & y_1 = y_0 + h f(x_0, y_0) = 2 + 1 \cdot \left(-\frac{0}{2}\right) = 2 \\ x_2 = 2 & y_2 = y_1 + h f(x_1, y_1) = 2 + 1 \cdot \left(-\frac{1}{2}\right) = \frac{3}{2} \end{array}$$

Comment. These are not good approximations! (To get better approximations, the step size must be chosen much smaller.) For comparison, the true values are $y(1) = \sqrt{3} \approx 1.73$ and $y(2) = 0$. Also note that we would get "bogus" values if we take another step to approximate $y(3)$ (whereas the true solution only exists until $x = 2$).

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Example 57. Consider the IVP $\frac{\mathrm{d}y}{\mathrm{d}x} = (2x - 3y)^2 + \frac{2}{3}$, $y(1) = \frac{1}{3}$.

(a) Approximate the solution $y(x)$ for $x \in [1, 2]$ using Euler's method with 2 steps.

(b) Approximate the solution $y(x)$ for $x \in [1, 2]$ using Euler's method with 3 steps.

(c) Solve this IVP exactly. Compare the values at $x = 2$.

Solution.

(a) The step size is $h = \frac{2-1}{2} = \frac{1}{2}$. We apply Euler's method with $f(x, y) = (2x - 3y)^2 + \frac{2}{3}$:

$$\begin{array}{ll} x_0 = 1 & y_0 = \frac{1}{3} \\ x_1 = \frac{3}{2} & y_1 = y_0 + h f(x_0, y_0) = \frac{1}{3} + \frac{1}{2} \cdot \left[\left(2 \cdot 1 - 3 \cdot \frac{1}{3}\right)^2 + \frac{2}{3}\right] = \frac{7}{6} \\ x_2 = 2 & y_2 = y_1 + h f(x_1, y_1) = \frac{7}{6} + \frac{1}{2} \cdot \frac{11}{12} = \frac{13}{8} \end{array}$$

In particular, the approximation for $y(2)$ is $y_2 = \frac{13}{8} = 1.625$.

(b) The step size is $h = \frac{2-1}{3} = \frac{1}{3}$. We again apply Euler's method with $f(x, y) = (2x - 3y)^2 + \frac{2}{3}$:

$$\begin{array}{ll} x_0 = 1 & y_0 = \frac{1}{3} \\ x_1 = \frac{4}{3} & y_1 = y_0 + h f(x_0, y_0) = \frac{1}{3} + \frac{1}{3} \cdot \left[\left(2 \cdot 1 - 3 \cdot \frac{1}{3}\right)^2 + \frac{2}{3}\right] = \frac{8}{9} \\ x_2 = \frac{5}{3} & y_2 = y_1 + h f(x_1, y_1) = \frac{8}{9} + \frac{1}{3} \cdot \frac{2}{3} = \frac{10}{9} \\ x_3 = 2 & y_3 = y_2 + h f(x_2, y_2) = \frac{10}{9} + \frac{1}{3} \cdot \frac{2}{3} = \frac{4}{3} \end{array}$$

In particular, the approximation for $y(2)$ is $y_3 = \frac{4}{3} \approx 1.333$.

(c) We solved this IVP in Example 38 using the substitution $u = 2x - 3y$ followed by separation of variables.
We found that the unique solution of the IVP is $y(x) = \frac{2}{3}x - \frac{1}{3(3x-2)}$.
In particular, the exact value at $x = 2$ is $y(2) = \frac{5}{4} = 1.25$.
We observe that our approximations for $y(2) = 1.25$ improved from 1.625 to 1.333 as we increased the number of steps (equivalently, we decreased the step size h from $\frac{1}{2}$ to $\frac{1}{3}$).
For comparison. With 10 steps (so that $h = \frac{1}{10}$), the approximation improves to $y(2) \approx 1.259$.

Spotlight on the exponential function

Example 58. Solve $y' = ky$ where k is a constant.

Solution. (experience) At this point, we can probably see that $y(x) = e^{kx}$ is a solution.
In fact, the general solution is $y(x) = Ce^{kx}$.
That there cannot be any further solutions follows from the existence and uniqueness theorem (see next example).

Solution. (separation of variables) Alternatively, we can solve the DE using separation of variables.
Express the DE as $\frac{\mathrm{d}y}{\mathrm{d}x} = ky$, then write it as $\frac{1}{y}\mathrm{d}y = k\,\mathrm{d}x$ (note that we just lost the solution $y = 0$).
Integrating gives $\ln|y| = kx + D$, hence $|y| = e^{kx+D}$.
Since the RHS is never zero, $y = \pm e^{kx+D} = Ce^{kx}$ (with $C = \pm e^D$). Finally, note that $C = 0$ corresponds to the singular solution $y = 0$ that we lost. In summary, the general solution is Ce^{kx}.

Example 59. Consider the IVP $y' = ky$, $y(a) = b$. Discuss existence and uniqueness of solutions.

Solution. The IVP is $y' = f(x, y)$ with $f(x, y) = ky$. We compute that $\frac{\partial}{\partial y} f(x, y) = k$.
We observe that both $f(x, y)$ and $\frac{\partial}{\partial y} f(x, y)$ are continuous for all (x, y).
Hence, for any initial conditions, the IVP locally has a unique solution by the existence and uniqueness theorem.

Comment. As a consequence, there can be no other solutions to the DE $y' = ky$ than the ones of the form $y(x) = Ce^{kx}$. Why?! [Assume that $y(x)$ satisfies $y' = ky$ and let (a, b) any value on the graph of y. Then $y(x)$ solves the IVP $y' = ky$, $y(a) = b$; but so does Ce^{kx} with $C = b/e^{ka}$. The uniqueness implies that $y(x) = Ce^{kx}$.]

In particular, we have the following characterization of the exponential function:

e^x is the unique solution to the IVP $y' = y$, $y(0) = 1$.

Comment. Note that, for instance, $\frac{\mathrm{d}}{\mathrm{d}x} 2^x = \ln(2)\, 2^x$. (This follows from $2^x = e^{\ln(2^x)} = e^{x\ln(2)}$.)
Since $\ln = \log_e$, this means that we cannot avoid the natural base $e \approx 2.718$ even if we try to use another base.

Euler's method applied to e^x

Example 60. Consider the IVP $y' = y$, $y(0) = 1$. Approximate the solution $y(x)$ for $x \in [0, 1]$ using Euler's method with 4 steps. In particular, what is the approximation for $y(1)$?

Comment. Of course, the real solution is $y(x) = e^x$. In particular, $y(1) = e \approx 2.71828$.

Solution. The step size is $h = \frac{1-0}{4} = \frac{1}{4}$. We apply Euler's method with $f(x, y) = y$:

$$\begin{array}{ll}
x_0 = 0 & y_0 = 1 \\
x_1 = \frac{1}{4} & y_1 = y_0 + h f(x_0, y_0) = 1 + \frac{1}{4} \cdot 1 = \frac{5}{4} = 1.25 \\
x_2 = \frac{1}{2} & y_2 = y_1 + h f(x_1, y_1) = \frac{5}{4} + \frac{1}{4} \cdot \frac{5}{4} = \frac{5^2}{4^2} = 1.5625 \\
x_3 = \frac{3}{4} & y_3 = y_2 + h f(x_2, y_2) = \frac{5^2}{4^2} + \frac{1}{4} \cdot \frac{5^2}{4^2} = \frac{5^3}{4^3} \approx 1.9531 \\
x_4 = 1 & y_4 = y_3 + h f(x_3, y_3) = \frac{5^3}{4^3} + \frac{1}{4} \cdot \frac{5^3}{4^3} = \frac{5^4}{4^4} \approx 2.4414
\end{array}$$

In particular, the approximation for $y(1)$ is $y_4 \approx 2.4414$.

Comment. Can you see that, if instead we start with $h = \frac{1}{n}$, then we similarly get $x_i = \frac{(n+1)^i}{n^i}$ for $i = 0, 1, \ldots, n$?
In particular, $y(1) \approx y_n = \frac{(n+1)^n}{n^n} = \left(1 + \frac{1}{n}\right)^n \to e$ as $n \to \infty$. Do you recall how to derive this final limit?

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Example 61. (cont'd) Consider the IVP $y' = y$, $y(0) = 1$. Approximate the solution $y(x)$ for $x \in [0, 1]$ using Euler's method with n steps for several values of n. In each case, what is the approximation for $y(1)$?

Solution. Since the real solution is $y(x) = e^x$ so that, in particular, the exact solution is $y(1) = e \approx 2.71828$. We proceed as we did in Example 60 in the case $n = 4$ and apply Euler's method with $f(x, y) = y$:

$$\begin{aligned} x_{n+1} &= x_n + h \\ y_{n+1} &= y_n + h \underbrace{f(x_n, y_n)}_{= y_n} = (1 + h) y_n \end{aligned}$$

We observe that it follows from $y_{n+1} = (1 + h) y_n$ that $y_n = (1 + h)^n y_0$. Since $y_0 = 1$ and $h = \frac{1 - 0}{n} = \frac{1}{n}$, we conclude that

$$x_n = 1, \quad y_n = \left(1 + \frac{1}{n}\right)^n.$$

[For instance, for $n = 4$, we get $x_4 = 1$, $y_4 = \left(\frac{5}{4}\right)^4 \approx 2.4414$ as in Example 60.]

In particular, our approximation for $y(1)$ is $\left(1 + \frac{1}{n}\right)^n$.

Here are a few values spelled out:

$$\begin{aligned} n = 1: &\quad \left(1 + \frac{1}{n}\right)^n = 2 \\ n = 4: &\quad \left(1 + \frac{1}{n}\right)^n = 2.4414\ldots \\ n = 12: &\quad \left(1 + \frac{1}{n}\right)^n = 2.6130\ldots \\ n = 100: &\quad \left(1 + \frac{1}{n}\right)^n = 2.7048\ldots \\ n = 365: &\quad \left(1 + \frac{1}{n}\right)^n = 2.7145\ldots \\ n = 1000: &\quad \left(1 + \frac{1}{n}\right)^n = 2.7169\ldots \\ n \to \infty: &\quad \left(1 + \frac{1}{n}\right)^n \to e = 2.71828\ldots \end{aligned}$$

We can see that Euler's method converges to the correct value as $n \to \infty$. On the other hand, we can see that it doesn't converge impressively fast. That is why, for serious applications, one usually doesn't use Euler's method directly but rather higher-order methods derived from the same principles (such as Runge–Kutta methods).

Interpretation. Note that we can interpret the above values in terms of compound interest. We start with initial capital of $y(0) = 1$ and we are interested in the capital $y(1)$ after 1 year if we receive interest at an annual rate of 100%:

- If we receive a single interest payment at the end of the year, then $y(1) = 2$ (case $n = 1$ above).
- If we receive quarterly interest payments of $\frac{100\%}{4} = 25\%$ each, then $y(1) = (1.25)^4 = 2.441\ldots$ (case $n = 4$).
- If we receive monthly interest payments of $\frac{100\%}{12} = \frac{1}{12}$ each, then $y(1) = 2.6130\ldots$ (case $n = 12$).
- If we receive daily interest payments of $\frac{100\%}{365} = \frac{1}{365}$ each, then $y(1) = 2.7145\ldots$ (case $n = 365$).

It is natural to wonder what happens if interest payments are made more and more frequently. Well, we already know the answer! If interest is compounded continuously, then we have e in our bank account after one year.

Challenge. Can you evaluate the limit $\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$ using your Calculus I skills?

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