Further entries in the Laplace transform table

Finally, we expand our table of Laplace transforms to the following:

$f(t)$	$F(s)$
$f'(t)$	$sF(s) - f(0)$
$f''(t)$	$s^2F(s) - sf(0) - f'(0)$
e^{at}	$\frac{1}{s-a}$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
t^n	$\frac{n!}{s^{n+1}}$
$e^{at}f(t)$	$F(s-a)$
$t\,f(t)$	$-F'(s)$
$u_a(t)\,f(t-a)$	$e^{-as}F(s)$

Example 160. (new entry) We add the following to our table of Laplace transforms:

$$\mathcal{L}(e^{at}f(t)) = \int_0^\infty e^{-st}e^{at}f(t)\mathrm{d}t = \int_0^\infty e^{-(s-a)t}f(t)\mathrm{d}t = F(s-a)$$

Example 161. (new entry) We also add the following to our table of Laplace transforms:

$$\mathcal{L}(t\,f(t)) = \int_0^\infty e^{-st}t\,f(t)\mathrm{d}t = \int_0^\infty -\frac{\mathrm{d}}{\mathrm{d}s}e^{-st}f(t)\mathrm{d}t = -\frac{\mathrm{d}}{\mathrm{d}s}\int_0^\infty e^{-st}f(t)\mathrm{d}t = -F'(s)$$

In particular,

$$\begin{aligned}
\mathcal{L}(t) &= \mathcal{L}(t\cdot 1) = -\frac{\mathrm{d}}{\mathrm{d}s}\frac{1}{s} = \frac{1}{s^2}\\
\mathcal{L}(t^2) &= -\frac{\mathrm{d}}{\mathrm{d}s}\frac{1}{s^2} = \frac{2}{s^3}\\
&\vdots\\
\mathcal{L}(t^n) &= \frac{n!}{s^{n+1}}.
\end{aligned}$$

Example 162. Determine the Laplace transform $\mathcal{L}((t-3)e^{2t})$.

Solution. $\mathcal{L}((t-3)e^{2t}) = \mathcal{L}(te^{2t}) - 3\mathcal{L}(e^{2t}) = \frac{1}{(s-2)^2} - \frac{3}{s-2}$

Here, we combined $\mathcal{L}(t\,f(t)) = -F'(s)$ with $\mathcal{L}(e^{2t}) = \frac{1}{s-2}$ to get $\mathcal{L}(te^{2t}) = -\frac{\mathrm{d}}{\mathrm{d}s}\frac{1}{s-2} = \frac{1}{(s-2)^2}$.

Alternative. Combine $\mathcal{L}(t-3) = \frac{1}{s^2} - \frac{3}{s}$ and $\mathcal{L}(f(t)e^{2t}) = F(s-2)$ to again get $\mathcal{L}((t-3)e^{2t}) = \frac{1}{(s-2)^2} - \frac{3}{s-2}$.

Example 163. Determine the inverse Laplace transform $\mathcal{L}^{-1}\left(\frac{1}{(s-3)^2}\right)$.

Solution. $\mathcal{L}^{-1}\left(\frac{1}{(s-3)^2}\right) = e^{3t}\mathcal{L}^{-1}\left(\frac{1}{s^2}\right) = te^{3t}$.

Example 164. Determine the inverse Laplace transform $\mathcal{L}^{-1}\left(\frac{e^{-2s}}{(s-3)^2}\right)$.

Solution. It follows from the previous example that $\mathcal{L}^{-1}\left(\frac{e^{-2s}}{(s-3)^2}\right) = u_2(t)(t-2)\,e^{3(t-2)}$.

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Example 165. Solve the IVP $y''-3y'+2y=e^t$, $y(0)=0$, $y'(0)=1$.

Solution. (old style, outline) The characteristic polynomial $D^2-3D+2=(D-1)(D-2)$. Since there is duplication, we have to look for a particular solution of the form $y_p=Ate^t$. To determine A, we need to plug into the DE (we find $A=-1$). Then, the general solution is $y(t)=Ate^t+C_1e^t+C_2e^{2t}$, and the initial conditions determine C_1 and C_2 (we find $C_1=-2$ and $C_2=2$).

Solution. (Laplace style)

$$\begin{aligned}
\mathcal{L}(y''(t))-3\mathcal{L}(y'(t))+2\mathcal{L}(y(t)) &= \mathcal{L}(e^t)\\
s^2Y(s)-sy(0)-y'(0)-3(sY(s)-y(0))+2Y(s) &= \frac{1}{s-1}\\
(s^2-3s+2)Y(s) &= 1+\frac{1}{s-1}=\frac{s}{s-1}\\
Y(s) &= \frac{s}{(s-1)^2(s-2)}
\end{aligned}$$

To find $y(t)$, we again use partial fractions. We find $Y(s)=\frac{A}{(s-1)^2}+\frac{B}{s-1}+\frac{C}{s-2}$ with coefficients (why?!)

$$C=\left.\frac{s}{(s-1)^2}\right|_{s=2}=2,\quad A=\left.\frac{s}{s-2}\right|_{s=1}=-1,\quad B=\left.\frac{\mathrm{d}}{\mathrm{d}s}\frac{s}{s-2}\right|_{s=1}=\left.\frac{-2}{(s-2)^2}\right|_{s=1}=-2.$$

Finally, $y(t)=\mathcal{L}^{-1}\left(\frac{A}{(s-1)^2}+\frac{B}{s-1}+\frac{C}{s-2}\right)=Ate^t+Be^t+Ce^{2t}=-(t+2)e^t+2e^{2t}$.

More details on the partial fractions with a repeated root. Above we computed A,B,C so that

$$\frac{s}{(s-1)^2(s-2)}=\frac{A}{(s-1)^2}+\frac{B}{s-1}+\frac{C}{s-2}.$$

- We can compute C as before by multiplying both sides with $s-2$ and then setting $s=2$.
- Similarly, we can compute A by multiplying both sides with $(s-1)^2$ and then setting $s=1$.
- To compute B, multiply both sides by $(s-1)^2$ (as for A) to get $\frac{s}{(s-2)}=A+B(s-1)+\frac{C(s-1)^2}{s-2}$.

  Now, we take the derivative on both sides (so that A goes away) to get

  $$\frac{-2}{(s-2)^2}=B+\frac{C(2(s-1)(s-2)-(s-1)^2)}{(s-2)^2}$$

  and we find B by setting $s=1$.

  Comment. In fact, the term involving C had to drop out when plugging in $s=1$, even after taking a derivative. That's because, after multiplying with $(s-1)^2$, that term has a double root at $s=1$. When taking a derivative, it therefore still has a (single) root at $s=1$.

Comment. A close relative to the Laplace transform is the **Fourier transform**:

$$F(\omega)=\int_{-\infty}^{\infty}e^{-i\omega t}f(t)\mathrm{d}t$$

Start with the Laplace transform and note that $s=\sigma+i\omega$ can be complex. If we focus on the purely imaginary case $\sigma=0$, and if $f(t)=0$ for $t<0$, then it turns into the Fourier transform.

We focused on the Laplace transform because it works particularly well for solving DEs. On the other hand, the Fourier transform is only defined if $f(t)$ decays sufficiently but works well for decomposing signals into their constituent frequencies.

Advanced. You may have also seen **Fourier series** which work for functions on a bounded interval $[-L,L]$ (or, equivalently, $2L$-periodic functions), in which case only a single frequency and its multiples appear, whereas the Fourier transform works for functions on the full real line (in a way, it is the limiting case $L\to\infty$).

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