Review. Underdamped, overdamped, critically damped for the DE $y'' + dy' + cy = 0$.

The critically damped case is when the damping is just large enough to avoid oscillations (underdamped case). If damping is increased further (overdamped case), then this typically means that it "takes longer" for the solution to approach equilibrium.

Challenge. Try to make this last comment more precise! For that, focus on the exponential term in the corresponding general solution that decays the slowest. That slowest term decays fastest in the critically damped case.

The same DE describes the current in an RLC circuit (disconnected from its source).

Application: motion of a pendulum

Example 105. Show that the motion of an ideal pendulum is described by

$$L\theta'' + g\sin(\theta) = 0,$$

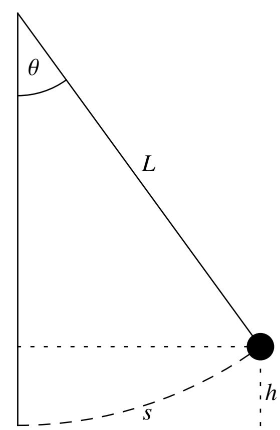
where θ is the angular displacement and L is the length of the pendulum.

And, as usual, g is acceleration due to gravity.

For short times and small angles, this motion is approximately described by

$$L\theta'' + g\theta = 0.$$

This is because, if θ is small, then $\sin(\theta) \approx \theta$. For instance, for $\theta = 15°$ the error $\theta - \sin\theta$ is about 1%.



Solution. (Newton's second law) The tangential component of the gravitational force is $F = -\sin\theta \cdot mg$. Combining this with Newton's second law, according to which $F = ma = mL\theta''$ (note that $a = s''$ where $s = L\theta$), we obtain the claimed DE.

Solution. (conservation of energy) Alternatively, we can use conservation of energy to derive the DE. Again, we assume the string to be massless, and let m be the swinging mass. Let s and h be as in the sketch above.

The velocity (more accurately, the speed) of the mass is $v = \frac{\mathrm{d}s}{\mathrm{d}t} = L\frac{\mathrm{d}\theta}{\mathrm{d}t}$.

Its kinetic energy therefore is $T = \frac{1}{2}mv^2 = \frac{1}{2}mL^2\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2$.

On the other hand, the potential energy is $V = mgh = mgL(1 - \cos\theta)$ (weight mg times height h).

By the principle of conservation of energy, the sum of these is constant: $T + V = \text{const}$

Taking the time derivative, this becomes $\frac{1}{2}mL^2 2\frac{\mathrm{d}\theta}{\mathrm{d}t}\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + mgL\sin\theta\frac{\mathrm{d}\theta}{\mathrm{d}t} = 0$. Cancelling terms, we obtain the DE.

Example 106. The motion of a pendulum is described by $\theta'' + 9\theta = 0$, $\theta(0) = 1/4$, $\theta'(0) = 0$. What is the period and the amplitude of the resulting oscillations?

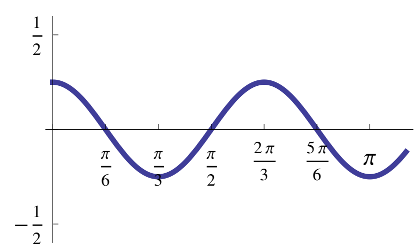
Solution. The roots of the characteristic polynomial are $\pm 3i$.

Hence, $\theta(t) = A\cos(3t) + B\sin(3t)$. $\theta(0) = A = 1/4$. $\theta'(0) = 3B = 0$.

Therefore, the solution is $\theta(t) = 1/4\cos(3t)$.

Hence, the period is $2\pi/3$ and the amplitude is $1/4$.

Comment. The initial angle $1/4$ is about $14.3°$.



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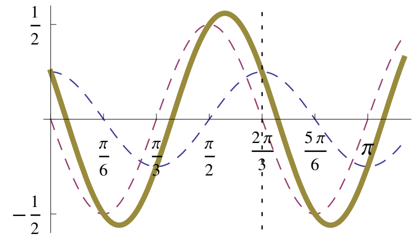
Example 107. The motion of a pendulum is described by $\theta'' + 9\theta = 0$, $\theta(0) = 1/4$, $\theta'(0) = -\frac{3}{2}$ ("initial kick"). What is the period and the amplitude of the resulting oscillations?

Solution. This time, $\theta(0) = A = 1/4$. $\theta'(0) = 3B = -3/2$.

Therefore, the solution is $\theta(t) = \frac{1}{4}\cos(3t) - \frac{1}{2}\sin(3t)$.

Hence, the period is $2\pi / 3$ and the amplitude is $\sqrt{\frac{1}{4^2} + \frac{1}{2^2}} = \frac{\sqrt{5}}{4} \approx 0.559$.

Comment. Using polar coordinates, we get $\theta(t) = \frac{\sqrt{5}}{4}\cos(3t - \alpha)$ with phase angle $\alpha = \tan^{-1}(-2) + 2\pi \approx 5.176$.



Adding external forces and the phenomenon of resonance

The motion of a mass m on a spring, with damping and with an external force $f(t)$ taken into account, can be modeled by the DE

$$my'' + dy' + ky = f(t).$$

Note that each term is representing a force: $my'' = ma$ is the force due to Newton's second law ($F = ma$), the term dy' models damping (proportional to the velocity), the term ky represents the force due to Hooke's law, and the term $f(t)$ represents an external force that acts on the mass at time t.

Example 108. Describe the solutions of $y'' + 4y = \cos(\lambda t)$. (Here, $\lambda > 0$ is a constant.)

Solution. The characteristic roots of the homogeneous DE are $\pm 2i$ so that 2 is the **natural frequency** (the frequency at which the system would oscillate in the absence of external forces; mathematically, this reflects the fact that the general solution to the corresponding homogeneous DE is $A\cos(2t) + B\sin(2t)$, which has frequency $\omega = 2$).

The characteristic roots of the inhomogeneous part are $\pm\lambda i$ where λ is the **external frequency**.

Case 1: $\boldsymbol{\lambda \neq 2}$. Then there is a particular solution of the form $y_p = A\cos(\lambda t) + B\sin(\lambda t)$. To determine the unique values of A, B, we plug into the DE:

$$y_p'' + 4y_p = (4 - \lambda^2)A\cos(\lambda t) + (4 - \lambda^2)B\sin(\lambda t) \overset{!}{=} \cos(\lambda t)$$

We conclude that $(4 - \lambda^2)A = 1$ and $(4 - \lambda^2)B = 0$. Solving these, we find $A = 1/(4 - \lambda^2)$ and $B = 0$.

Thus, the general solution is of the form $y = \frac{1}{4 - \lambda^2}\cos(\lambda t) + C_1\cos(2t) + C_2\sin(2t)$.

Case 2: $\boldsymbol{\lambda = 2}$. Now, there is a particular solution of the form $y_p = At\cos(2t) + Bt\sin(2t)$. To determine the unique values of A, B, we again plug into the DE (which is more work this time):

$$y_p'' + 4y_p \overset{\text{work}}{=} 4B\cos(2t) - 4A\sin(2t) \overset{!}{=} \cos(2t)$$

We conclude that $4B = 1$ and $-4A = 0$. Solving these, we find $A = 0$ and $B = 1/4$.

Thus, the general solution is of the form $y = \frac{1}{4}t\sin(2t) + C_1\cos(2t) + C_2\sin(2t)$.

Note that the amplitude in y_p increases without bound (so that the same is true for the general solution).

This phenomenon is called **resonance**; it occurs if an external frequency matches a natural frequency.

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If an external frequency matches a natural frequency, then **resonance** occurs.

In that case, we obtain amplitudes that grow without bound (as illustrated in Example 108).

Resonance (or anything close to it) is very important for practical purposes because large amplitudes can be very destructive: singing to shatter glass, earth quake waves and buildings, marching soldiers on bridges, ...

Comment. Mathematically speaking, resonance occurs if the characteristic roots of the homogeneous DE and the inhomogeneous part overlap. In that case, the solutions acquire a factor of the variable t (or x) which changes the nature of the solutions (and results in unbounded amplitudes).

Example 109. Consider $y'' + 9y = 10\cos(2\lambda t)$. For what value of λ does resonance occur?

Solution. The natural frequency is 3. The external frequency is 2λ. Hence, resonance occurs when $\lambda = \frac{3}{2}$.

Example 110. The motion of a mass on a spring under an external force is described by $5y'' + 2y = -2\sin(3\lambda t)$. For which value of λ does resonance occur?

Solution. The natural frequency is $\sqrt{\frac{2}{5}}$. The external frequency is 3λ. Hence, resonance occurs when $\lambda = \frac{1}{3}\sqrt{\frac{2}{5}}$.

Example 111. The motion of a mass on a spring under an external force is described by $3y'' + ky = \cos(t/2)$. For which value of $k > 0$ does resonance occur?

Solution. The natural frequency is $\sqrt{\frac{k}{3}}$. The external frequency is $\frac{1}{2}$. Hence, resonance occurs when $\sqrt{\frac{k}{3}} = \frac{1}{2}$. This happens if $k = 3 \cdot \left(\frac{1}{2}\right)^2 = \frac{3}{4}$.

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