The qualitative effects of damping

The motion of a mass on a spring (or the approximate motion of a pendulum), with damping taken into account, can be modeled by the DE

$$y'' + dy' + cy = 0$$

with $c > 0$ and $d \geqslant 0$. The term dy' models damping (e.g. friction, air resistance) proportional to the velocity y'.

The characteristic equation $r^2 + dr + c = 0$ has roots $\frac{1}{2}\left(-d \pm \sqrt{d^2 - 4c}\right)$.

The nature of the solutions depends on whether the **discriminant** $\Delta = d^2 - 4c$ is positive, negative, or zero.

Undamped. $d = 0$. In that case, $\Delta < 0$. We get two complex roots $\pm i\omega$ with $\omega = \sqrt{c}$.

Solutions: $A\cos(\omega t) + B\sin(\omega t) = r\cos(\omega t - \alpha)$ where $(A, B) = r(\cos\alpha, \sin\alpha)$
These are oscillations with frequency ω and amplitude r.

Underdamped. $d > 0$, $\Delta < 0$. We get two complex roots $-\rho \pm i\omega$ with $-\rho = -d/2 < 0$.

Solutions: $e^{-\rho t}[A\cos(\omega t) + B\sin(\omega t)] = e^{-\rho t}[r\cos(\omega t - \alpha)]$ ($\to 0$ as $t \to \infty$)
These are oscillations with amplitude going to zero.

Critically damped. $d > 0$, $\Delta = 0$. We get one (double) real root $-\rho < 0$.

Solutions: $(A + Bt)\,e^{-\rho t}$ ($\to 0$ as $t \to \infty$)
There are no oscillations. (Can you see why we cross the t-axis at most once?)

Overdamped. $d > 0$, $\Delta > 0$. We get two real roots $-\rho_1, -\rho_2 < 0$. [negative because $c, d > 0$]

Solutions: $A e^{-\rho_1 t} + B e^{-\rho_2 t}$ ($\to 0$ as $t \to \infty$)
There are no oscillations. (Again, there is at most one crossing of the t-axis.)

Example 103. The motion of a mass on a spring is described by $5y'' + dy' + 2y = 0$ with $d > 0$. For which value of d is the motion critically damped? Underdamped? Overdamped?

Solution. The characteristic roots are $\frac{1}{2}\left(-d \pm \sqrt{d^2 - 40}\right)$. The motion is critically damped if $d^2 - 40 = 0$. Equivalently, the motion is critically damped $d = \sqrt{40}$.

Consequently, the motion is underdamped if $d < \sqrt{40}$ (then we get complex roots and the solutions will involve oscillations), and it is overdamped if $d > \sqrt{40}$ (the roots are real and the solutions will not involve oscillations).

Example 104. The motion of a mass on a spring is described by $my'' + 3y' + 2y = 0$. For which value of m is the motion critically damped? Underdamped? Overdamped?

Solution. The characteristic roots are $\frac{1}{2}(-3 \pm \sqrt{9 - 8m})$. The motion is critically damped if $9 - 8m = 0$. Equivalently, the motion is critically damped $m = \frac{9}{8}$.

Consequently, the motion is underdamped if $m > \frac{9}{8}$ (then we get complex roots and the solutions will involve oscillations), and it is overdamped if $m < \frac{9}{8}$ (the roots are real and the solutions will not involve oscillations).

Armin Straub
straub@southalabama.edu