A closer look at second-order linear DEs

Application: motion of a mass on a spring

Example 99. The motion of a mass m attached to a spring is described by

$$my'' + ky = 0$$

where y is the displacement from the equilibrium position and $k > 0$ is the spring constant.

Why? This follows from Newton's second law $F = ma = my''$ combined with Hooke's law $F = -ky$. (Note that the minus sign is needed because the force on the mass is in direction opposite to the displacement.)

Comment. By measuring y as the displacement from equilibrium, it doesn't matter whether the mass is attached horizontally or vertically (gravity is taken into account by the extra stretch in the spring due to the mass).

Solving this DE, we find that the general solution is

$$y(t) = A\cos(\omega t) + B\sin(\omega t)$$

where $\omega = \sqrt{k/m}$ (note that the characteristic roots are $\pm i\sqrt{\frac{k}{m}}$). We observe that:

- The motion $y(t)$ is periodic with **period** $2\pi/\omega$.

  This follows from the fact that both $\cos(t)$ and $\sin(t)$ have period 2π.

- The **amplitude** of the motion $y(t)$ is $\sqrt{A^2 + B^2}$.

  This follows from the fact that $y(t) = A\cos(\omega t) + B\sin(\omega t) = r\cos(\omega t - \alpha)$ where (r, α) are the **polar coordinates** for (A, B). In particular, the amplitude is $r = \sqrt{A^2 + B^2}$. More below!

(period and amplitude of oscillations) The oscillations $A\cos(\omega t) + B\sin(\omega t)$ are periodic with **period** $2\pi/\omega$ and **amplitude** $r = \sqrt{A^2 + B^2}$.
More precisely, if (r, α) are the **polar coordinates** for (A, B), then

$$A\cos(\omega t) + B\sin(\omega t) = r\cos(\omega t - \alpha).$$

ω is the (circular) **frequency** and α is called the **phase angle**.

Why? First, observe that both sides of $A\cos(\omega t) + B\sin(\omega t) = r\cos(\omega t - \alpha)$ solve the same DE

$$y'' + \omega y = 0.$$

The LHS has initial values $y(0) = A$ and $y'(0) = \omega B$, the RHS has $y(0) = r\cos(\alpha)$ and $y'(0) = r\omega\sin(\alpha)$. Hence, the two are equal if $A = r\cos(\alpha)$ and $B = r\sin(\alpha)$.

Alternatively. If you like trig identities, this follows from:
$A\cos(\omega t) + B\sin(\omega t) = r(\cos(\alpha)\cos(\omega t) + \sin(\alpha)\sin(\omega t)) = r\cos(\omega t - \alpha)$.

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Review. How to calculate the polar coordinates (r, α) for (A, B)?

We need to find $r \geqslant 0$ and $\alpha \in [0, 2\pi)$ such that $(A, B) = r(\cos\alpha, \sin\alpha)$.

Hence, $r = \sqrt{A^2 + B^2}$ and α is determined by $\cos(\alpha) = \frac{A}{r}$ and $\sin(\alpha) = \frac{B}{r}$.

In particular, $\tan(\alpha) = \frac{B}{A}$ and, if careful, we can compute α using $\tan^{-1}$ as

$\alpha = \tan^{-1}\left(\frac{B}{A}\right) + \begin{cases} 0, & \text{if } A > 0, \\ \pi, & \text{if } A < 0. \end{cases}$

Comment. $\tan^{-1}$ gives angles between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ (corresponding to the first and fourth quadrants where $A > 0$).

Comment. If we want positive angles, we can always add 2π to negative angles.

Comment. In most programming languages, we can use the function $\operatorname{atan2}(B, A)$ to compute the corresponding angle α (between $-\pi$ and π).

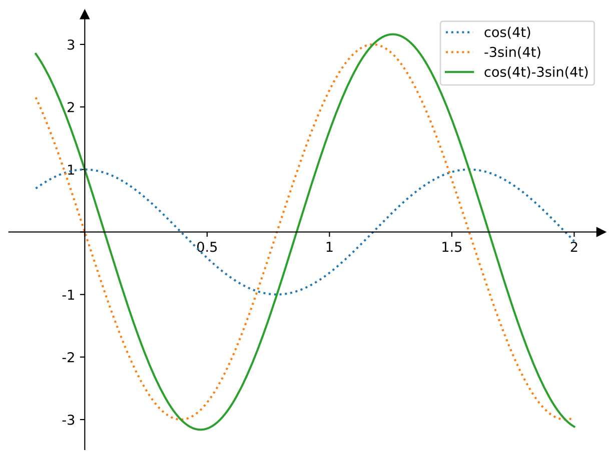
Example 100. What are the period, the amplitude and the phase angle of the oscillations $\cos(4t) - 3\sin(4t)$?

Solution. The period is $\frac{2\pi}{4} = \frac{\pi}{2}$.

The amplitude is $\sqrt{1^2 + (-3)^2} = \sqrt{10} \approx 3.16$.

To write $\cos(4t) - 3\sin(4t) = r\cos(4t - \alpha)$, we need to compute the polar coordinates (r, α) of the point $(1, -3)$. We already computed $r = \sqrt{10}$. The phase angle is $\alpha = \operatorname{atan2}(-3, 1) = \tan^{-1}(-3) \approx -1.249 \approx -71.6°$.

Comment. If we prefer positive angles, we can choose $\alpha = \tan^{-1}(-3) + 2\pi \approx 5.034 \approx 288.4°$ instead.



Example 101. What are the period, the amplitude and the phase angle of the oscillations $-\cos(4t) + 3\sin(4t)$?

Solution. Again, the period is $\frac{2\pi}{4} = \frac{\pi}{2}$ and the amplitude is still $\sqrt{(-1)^2 + 3^2} = \sqrt{10}$.

This time, to write $-\cos(4t) + 3\sin(4t) = r\cos(4t - \alpha)$, we need to compute the polar coordinates (r, α) of the point $(-1, 3)$. We already computed $r = \sqrt{10}$. The phase angle is $\alpha = \operatorname{atan2}(3, -1) = \tan^{-1}(-3) + \pi \approx 1.893 \approx 108.4°$.

Comment. Of course, this example is the same as the previous times -1. In particular, make a sketch to compare the polar coordinates of $(1, -3)$ and $(-1, 3)$, and to make sure you see why they differ by exactly $180°$.

Example 102. The motion of a mass on a spring is described by $5y'' + 2y = 0$, $y(0) = 3$, $y'(0) = -1$. What is the period and the amplitude of the resulting oscillations?

Solution. The characteristic roots are $\pm i\omega$ with $\omega = \sqrt{\frac{2}{5}}$. The general solution is $y(t) = A\cos(\omega t) + B\sin(\omega t)$.

The period of the oscillations therefore is $\frac{2\pi}{\omega} = 2\pi\sqrt{\frac{5}{2}} = \pi\sqrt{10}$.

For the initial conditions, we need $y(0) = A \overset{!}{=} 3$ and $y'(0) = \omega B \overset{!}{=} -1$ (since $y'(t) = -\omega A\sin(\omega t) + \omega B\cos(\omega t)$).

Hence, $A = 3$ and $B = -\frac{1}{\omega} = -\sqrt{\frac{5}{2}}$.

The amplitude of the oscillations is $\sqrt{A^2 + B^2} = \sqrt{3^2 + \frac{5}{2}} = \sqrt{\frac{23}{2}}$.