Example 93. Consider the DE $y'' - 2y' + y = 5\sin(3x)$.

(a) What is the simplest form (with undetermined coefficients) of a particular solution?

(b) Determine a particular solution.

(c) Determine the general solution.

Solution. Note that $D^2 - 2D + 1 = (D-1)^2$.

	homogeneous DE	inhomogeneous part
characteristic roots	$1, 1$	$\pm 3i$
solutions	$e^x, x e^x$	$\cos(3x), \sin(3x)$

(a) This tells us that there exists a particular solution of the form $y_p = A\cos(3x) + B\sin(3x)$.

(b) To find the values of A and B, we plug into the DE.
$y_p' = -3A\sin(3x) + 3B\cos(3x)$
$y_p'' = -9A\cos(3x) - 9B\sin(3x)$
$y_p'' - 2y_p' + y_p = (-8A - 6B)\cos(3x) + (6A - 8B)\sin(3x) \overset{!}{=} 5\sin(3x)$
Equating the coefficients of $\cos(x)$, $\sin(x)$, we obtain the two equations $-8A - 6B = 0$ and $6A - 8B = 5$.
Solving these, we find $A = \frac{3}{10}$, $B = -\frac{2}{5}$. Accordingly, a particular solution is $y_p = \frac{3}{10}\cos(3x) - \frac{2}{5}\sin(3x)$.

(c) The general solution is $y(x) = \frac{3}{10}\cos(3x) - \frac{2}{5}\sin(3x) + (C_1 + C_2 x)e^x$.

Example 94. Consider the DE $y'' - 2y' + y = 5e^{2x}\sin(3x) + 7x e^x$. What is the simplest form (with undetermined coefficients) of a particular solution?

Solution. Since $D^2 - 2D + 1 = (D-1)^2$, the characteristic roots are $1, 1$. The roots for the inhomogeneous part are $2 \pm 3i, 1, 1$. Hence, there has to be a particular solution of the form $y_p = A_1 e^{2x}\cos(3x) + A_2 e^{2x}\sin(3x) + A_3 x^2 e^x + A_4 x^3 e^x$.
(We can then plug into the DE to determine the (unique) values of the coefficients A_1, A_2, A_3, A_4.)

Example 95. (homework) What is the shape of a particular solution of $y'' + 4y' + 4y = x\cos(x)$?

Solution. The characteristic roots are $-2, -2$. The roots for the inhomogeneous part are $\pm i, \pm i$. Hence, there has to be a particular solution of the form $y_p = (A_1 + A_2 x)\cos(x) + (A_3 + A_4 x)\sin(x)$.

Continuing to find a particular solution. To find the value of the A_j's, we plug into the DE.
$y_p' = (A_2 + A_3 + A_4 x)\cos(x) + (A_4 - A_1 - A_2 x)\sin(x)$
$y_p'' = (2A_4 - A_1 - A_2 x)\cos(x) + (-2A_2 - A_3 - A_4 x)\sin(x)$
$$y_p'' + 4y_p' + 4y_p = (3A_1 + 4A_2 + 4A_3 + 2A_4 + (3A_2 + 4A_4)x)\cos(x) + (-4A_1 - 2A_2 + 3A_3 + 4A_4 + (-4A_2 + 3A_4)x)\sin(x) \overset{!}{=} x\cos(x).$$
Equating the coefficients of $\cos(x)$, $x\cos(x)$, $\sin(x)$, $x\sin(x)$, we get the equations $3A_1 + 4A_2 + 4A_3 + 2A_4 = 0$, $3A_2 + 4A_4 = 1$, $-4A_1 - 2A_2 + 3A_3 + 4A_4 = 0$, $-4A_2 + 3A_4 = 0$.
Solving (this is tedious!), we find $A_1 = -\frac{4}{125}$, $A_2 = \frac{3}{25}$, $A_3 = -\frac{22}{125}$, $A_4 = \frac{4}{25}$.
Hence, $y_p = \left(-\frac{4}{125} + \frac{3}{25}x\right)\cos(x) + \left(-\frac{22}{125} + \frac{4}{25}x\right)\sin(x)$.

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Example 96. (homework) What is the shape of a particular solution of $y'' + 4y' + 4y = 4e^{3x}\sin(2x) - x\sin(x)$?

Solution. The characteristic roots are $-2, -2$. The roots for the inhomogeneous part roots are $3 \pm 2i, \pm i, \pm i$. Hence, there has to be a particular solution of the form

$y_p = A_1 e^{3x}\cos(2x) + A_2 e^{3x}\sin(2x) + (A_3 + A_4 x)\cos(x) + (A_5 + A_6 x)\sin(x)$.

Continuing to find a particular solution. To find the values of $A_1, \ldots, A_6$, we plug into the DE. But this final step is so boring that we don't go through it here. Computers (currently?) cannot afford to be as selective; mine obediently calculated: $y_p = -\frac{4}{841}e^{3x}(20\cos(2x) - 21\sin(2x)) + \frac{1}{125}((-22 + 20x)\cos(x) + (4 - 15x)\sin(x))$

Excursion: Euler's identity

Let's revisit Euler's identity from Theorem 83.

Theorem 97. (Euler's identity) $e^{ix} = \cos(x) + i\sin(x)$

Proof. Observe that both sides are the (unique) solution to the IVP $y' = iy$, $y(0) = 1$.
[Check that by computing the derivatives and verifying the initial condition! As we did in class.] □

On lots of T-shirts. In particular, with $x = \pi$, we get $\boxed{e^{\pi i} = -1}$ or $e^{i\pi} + 1 = 0$ (which connects the five fundamental constants).

Example 98. Where do trig identities like $\sin(2x) = 2\cos(x)\sin(x)$ or $\sin^2(x) = \frac{1 - \cos(2x)}{2}$ (and infinitely many others!) come from?

Short answer: they all come from the simple exponential law $e^{x+y} = e^x e^y$.

Let us illustrate this in the simple case $(e^x)^2 = e^{2x}$. Observe that

$$\begin{aligned} e^{2ix} &= \cos(2x) + i\sin(2x) \\ e^{ix}e^{ix} &= [\cos(x) + i\sin(x)]^2 = \cos^2(x) - \sin^2(x) + 2i\cos(x)\sin(x). \end{aligned}$$

Comparing imaginary parts (the "stuff with an i"), we conclude that $\sin(2x) = 2\cos(x)\sin(x)$.
Likewise, comparing real parts, we read off $\cos(2x) = \cos^2(x) - \sin^2(x)$.

(Use $\cos^2(x) + \sin^2(x) = 1$ to derive $\sin^2(x) = \frac{1 - \cos(2x)}{2}$ from the last equation.)

Challenge. Can you find a triple-angle trig identity for $\cos(3x)$ and $\sin(3x)$ using $(e^x)^3 = e^{3x}$?
Or, use $e^{i(x+y)} = e^{ix}e^{iy}$ to derive $\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$ and $\sin(x+y) = \ldots$

Realize that the complex number $e^{i\theta} = \cos(\theta) + i\sin(\theta)$ corresponds to the point $(\cos(\theta), \sin(\theta))$. These are precisely the points on the unit circle!

Recall that a point (x, y) can be represented using **polar coordinates** (r, θ), where r is the distance to the origin and θ is the angle with the x-axis.

Then, $x = r\cos\theta$ and $y = r\sin\theta$.

Every complex number z can be written in **polar form** as $z = re^{i\theta}$, with $r = |z|$.

Why? By comparing with the usual polar coordinates ($x = r\cos\theta$ and $y = r\sin\theta$), we can write

$$z = x + iy = r\cos\theta + ir\sin\theta = re^{i\theta}.$$

In the final step, we used Euler's identity.

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