**Example 40.** Solve $(x-y)\frac{dy}{dx} = x+y$.

**Solution.** Divide the DE by $x$ to get $\left(1-\frac{y}{x}\right)\frac{dy}{dx} = 1+\frac{y}{x}$. This is a DE of the form $y' = F\left(\frac{y}{x}\right)$.

We therefore substitute $u = \frac{y}{x}$. Then $y = ux$ and $\frac{dy}{dx} = x\frac{du}{dx} + u$.

The resulting DE is $(x-ux)\left(x\frac{du}{dx}+u\right) = x+ux$, which simplifies to $x(1-u)\frac{du}{dx} = 1+u^2$.

This DE is separable: $\frac{1-u}{1+u^2}du = \frac{1}{x}dx$

Integrating both sides, we find $\arctan(u) - \frac{1}{2}\ln(1+u^2) = \ln|x| + C$.

Setting $u = y/x$, we get the (general) implicit solution $\arctan(y/x) - \frac{1}{2}\ln(1+(y/x)^2) = \ln|x| + C$.

**Comment.** We used $\int \frac{1}{1+u^2}du = \arctan(u) + C$ and $\int \frac{x}{1+x^2}dx = \frac{1}{2}\ln(1+x^2) + C$ when integrating.

See Example 36 where we reviewed these integrals.

**Example 41.** Solve the IVP $\frac{dy}{dx} = 2y - 3xy^5$, $y(0) = 1$.

**Solution.** This is an example of a Bernoulli equation (with $n=5$). We therefore substitute $u = y^{1-n} = y^{-4}$.

Accordingly, $y = u^{-1/4}$ and, thus, $\frac{dy}{dx} = -\frac{1}{4}u^{-5/4}\frac{du}{dx}$.

The new DE is $-\frac{1}{4}u^{-5/4}\frac{du}{dx} = 2u^{-1/4} - 3xu^{-5/4}$, which simplifies to $\frac{du}{dx} = -8u + 12x$.

This is a linear first-order DE, which we solve according to our recipe:

(a) Rewrite the DE as $\frac{du}{dx} + P(x)u = Q(x)$ with $P(x) = 8$ and $Q(x) = 12x$.

(b) The integrating factor is $f(x) = \exp\left(\int P(x)dx\right) = e^{8x}$.

(c) Multiply the (rewritten) DE by $f(x) = e^{8x}$ to get

$$\underbrace{e^{8x}\frac{du}{dx} + 8e^{8x}u}_{=\frac{d}{dx}[e^{8x}u]} = 12xe^{8x}.$$

(d) Integrate both sides to get:

$$e^{8x}u = 12\int xe^{8x}dx = 12\left(\frac{1}{8}xe^{8x} - \frac{1}{8^2}e^{8x}\right) + C = \frac{3}{2}xe^{8x} - \frac{3}{16}e^{8x} + C$$

Here we used that $\int xe^{ax}dx = \frac{1}{a}xe^{ax} - \frac{1}{a^2}e^{ax}$. (Integration by parts!)

The general solution of the DE for $u$ therefore is $u = \frac{3}{2}x - \frac{3}{16} + Ce^{-8x}$.

Correspondingly, the general solution of the initial DE is $y = u^{-1/4} = 1/\sqrt[4]{\frac{3}{2}x - \frac{3}{16} + Ce^{-8x}}$.

Using $y(0) = 1$, we find $1 = 1/\sqrt[4]{C - \frac{3}{16}}$ from which we obtain $C = 1 + \frac{3}{16} = \frac{19}{16}$.

The unique solution to the IVP therefore is $y = 1/\sqrt[4]{\frac{3}{2}x - \frac{3}{16} + \frac{19}{16}e^{-8x}}$.


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## Solving simple 2nd order DEs

We have the following two useful substitutions for certain simple DEs of order $2$:

- $F(y'', y', x) = 0$ (2nd order with "$y$ missing")

  Set $u = y' = \frac{\mathrm{d}y}{\mathrm{d}x}$. Then $y'' = \frac{\mathrm{d}u}{\mathrm{d}x}$. We get the first-order DE $F\left(\frac{\mathrm{d}u}{\mathrm{d}x}, u, x\right) = 0$.

- $F(y'', y', y) = 0$ (2nd order with "$x$ missing")

  Set $u = y' = \frac{\mathrm{d}y}{\mathrm{d}x}$. Then $y'' = \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}y} \cdot \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}y} \cdot u$. We get the first-order DE $F\left(u\frac{\mathrm{d}u}{\mathrm{d}y}, u, y\right) = 0$.

**Example 42.** Solve $y'' = x - y'$.

**Solution.** We substitute $u = y'$, which results in the first-order DE $u' = x - u$.
This DE is linear and, using our recipe (see below for the details), we can solve it to find $u = x - 1 + Ce^{-x}$.
Since $y' = u$, we conclude that the general solution is $y = \int (x - 1 + Ce^{-x})\mathrm{d}x = \frac{1}{2}x^2 - x - Ce^{-x} + D$.

**Important comment.** This is a DE of order $2$. Hence, as expected, the general solution has two free parameter.

**Solving the linear DE.** To solve $u' = x - u$ (also see Example 32, where we had solved this DE before), we

(a) rewrite the DE as $\frac{\mathrm{d}u}{\mathrm{d}x} + P(x)u = Q(x)$ with $P(x) = 1$ and $Q(x) = x$.

(b) The integrating factor is $f(x) = \exp\left(\int P(x)\mathrm{d}x\right) = e^x$.

(c) Multiply the (rewritten) DE by $f(x) = e^x$ to get $\underbrace{e^x\frac{\mathrm{d}u}{\mathrm{d}x} + e^x u}_{= \frac{\mathrm{d}}{\mathrm{d}x}[e^x u]} = xe^x$.

(d) Integrate both sides to get (using integration by parts): $e^x u = \int xe^x \mathrm{d}x = xe^x - e^x + C$

Hence, the general solution of the DE for $u$ is $u = x - 1 + Ce^{-x}$, which is what we used above.

**Example 43. (homework)** Solve the IVP $y'' = x - y'$, $y(0) = 1$, $y'(0) = 2$.

**Solution.** As in the previous example, we find that the general solution to the DE is $y(x) = \frac{1}{2}x^2 - x - Ce^{-x} + D$.
Using $y'(x) = x - 1 + Ce^{-x}$ and $y'(0) = 2$, we find that $2 = -1 + C$. Hence, $C = 3$.
Then, using $y(x) = \frac{1}{2}x^2 - x - 3e^{-x} + D$ and $y(0) = 1$, we find $1 = -3 + D$. Hence, $D = 4$.
In conclusion, the unique solution to the IVP is $y(x) = \frac{1}{2}x^2 - x - 3e^{-x} + 4$.

**Example 44. (extra)** Find the general solution to $y'' = 2yy'$.

**Solution.** We substitute $u = y' = \frac{\mathrm{d}y}{\mathrm{d}x}$. Then $y'' = \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}y} \cdot \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}y} \cdot u$.
Therefore, our DE turns into $u\frac{\mathrm{d}u}{\mathrm{d}y} = 2yu$.
Dividing by $u$, we get $\frac{\mathrm{d}u}{\mathrm{d}y} = 2y$. [Note that we lose the solution $u = 0$, which gives the singular solution $y = C$.]
Hence, $u = y^2 + C$. It remains to solve $y' = y^2 + C$. This is a separable DE.
$\frac{1}{C + y^2}\mathrm{d}y = \mathrm{d}x$. Let us restrict to $C = D^2 \geqslant 0$ here. (This means we will only find "half" of the solutions.)
$\int \frac{1}{D^2 + y^2}\mathrm{d}y = \frac{1}{D^2}\int \frac{1}{1 + (y/D)^2}\mathrm{d}y = \frac{1}{D}\arctan(y/D) = x + A$.
Solving for $y$, we find $y = D\tan(Dx + AD) = D\tan(Dx + B)$. $[B = AD]$

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