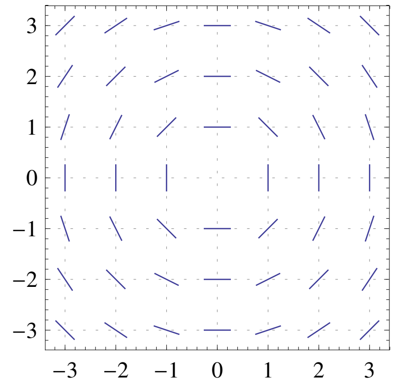
Slope fields, or sketching solutions to DEs

The next example illustrates that we can "plot" solutions to differential equations (it does not matter if we are able to actually solve them).

Comment. This is an important point because "plotting" really means that we can numerically approximate solutions. For complicated systems of differential equations, such as those used to model fluid flow, this is usually the best we can do. Nobody can actually solve these equations.

Example 13. Consider the DE $y' = -x/y$.

Let's pick a point, say, $(1,2)$. If a solution $y(x)$ is passing through that point, then its slope has to be $y' = -1/2$. We therefore draw a small line through the point $(1,2)$ with slope $-1/2$. Continuing in this fashion for several other points, we obtain the **slope field** on the right.



With just a little bit of imagination, we can now anticipate the solutions to look like (half)circles around the origin. Let us check whether $y(x) = \sqrt{r^2 - x^2}$ might indeed be a solution!

$y'(x) = \frac{1}{2}\frac{-2x}{\sqrt{r^2 - x^2}} = -x/y(x)$. So, yes, we actually found solutions!

Solving DEs: Separation of variables

Example 14. Solve the DE $y' = -\frac{x}{y}$.

Solution. Rewrite the DE as $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}$.

Separate the variables to get $y\,\mathrm{d}y = -x\,\mathrm{d}x$ (in particular, we are multiplying both sides by $\mathrm{d}x$).

Integrating both sides, we get $\int y\,\mathrm{d}y = \int -x\,\mathrm{d}x$.

Computing both integrals results in $\frac{1}{2}y^2 = -\frac{1}{2}x^2 + C$ (we combine the two constants of integration into one).

Hence $x^2 + y^2 = D$ (with $D = 2C$).

This is an **implicit form** of the solutions to the DE. We can make it explicit by solving for y. Doing so, we find $y(x) = \pm\sqrt{D - x^2}$ (choosing $+$ gives us the upper half of a circle, while the negative sign gives us the lower half).

Comment. The step above where we break $\frac{\mathrm{d}y}{\mathrm{d}x}$ apart and then integrate may sound sketchy!

However, keep in mind that, after we find a solution $y(x)$, even if by sketchy means, we can (and should!) verify that $y(x)$ is indeed a solution by plugging into the DE. We actually already did that in the previous example!

Example 15. Solve the IVP $y' = -\frac{x}{y}$, $y(0) = -3$.

Comment. Instead of using what we found in Example 14, we start from scratch to better illustrate the solution process (and how to use the initial condition right away to determine the value of the constant of integration).

Solution. We separate variables to get $y\,\mathrm{d}y = -x\,\mathrm{d}x$.

Integrating gives $\frac{1}{2}y^2 = -\frac{1}{2}x^2 + C$, and we use $y(0) = -3$ to find $\frac{1}{2}(-3)^2 = 0 + C$ so that $C = \frac{9}{2}$.

Hence, $x^2 + y^2 = 9$ is an **implicit** form of the solution.

Solving for y, we get $y = -\sqrt{9 - x^2}$ (note that we have to choose the negative sign so that $y(0) = -3$).

Comment. Note that our solution is a **local solution**, meaning that it is valid (and solves the DE) locally around $x = 0$ (from the initial condition). However, it is not a **global solution** because it doesn't make sense outside of x in the interval $[-3, 3]$.

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Which differential equations can we actually solve using separation of variables?

- A general DE of first order is typically of the form $\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y)$.

  For instance, $\frac{\mathrm{d}y}{\mathrm{d}x} = \sin(xy) - x^2 y$.

  Comment. First order means that only the first derivative of y shows up. The most general form of a DE of first order is $F(x, y, y') = 0$ but we can usually solve for y' to get to the above form.

- The ones we can solve are **separable equations**, which are of the form $\frac{\mathrm{d}y}{\mathrm{d}x} = g(x)h(y)$.

  Example. The equation $\frac{\mathrm{d}y}{\mathrm{d}x} = y - x$ (although simple) is not separable.

  Example. The equation $\frac{\mathrm{d}y}{\mathrm{d}x} = e^{y-x}$ is separable because we can write it as $\frac{\mathrm{d}y}{\mathrm{d}x} = e^y e^{-x}$.

Example 16. (extra)

Comment. In this example, we use $x(t)$ instead of $y(x)$ for the function described by the differential equation. In general, of course, any choice of variable names is possible. If we write something like x' or y' it needs to be clear from the context with respect to which variable that derivative is meant (such as $x' = \frac{\mathrm{d}}{\mathrm{d}t} x(t)$).

(a) Solve the DE $\frac{\mathrm{d}x}{\mathrm{d}t} = kx^2$.

(b) Verify your answer from the first part.

(c) Solve the IVP $\frac{\mathrm{d}x}{\mathrm{d}t} = kx^2$, $x(0) = 2$.

(d) Solve the IVP $\frac{\mathrm{d}x}{\mathrm{d}t} = kx^2$, $x(0) = 0$.

Solution.

(a) This DE is separable: $\frac{1}{x^2}\mathrm{d}x = k\,\mathrm{d}t$. Integrating, we find $-\frac{1}{x} = kt + B$. (We plan to replace B by a new constant C in a moment.) Hence, $x = -\frac{1}{kt + B} = \frac{1}{C - kt}$.

[Here, $C = -B$ but that relationship doesn't matter; it only matters that the solution has a free parameter.]

Comment. Note that we did not find the solution $x = 0$ (lost when dividing by x^2). It is called a **singular solution** because it is not part of the **general solution** (the one-parameter family found above). [Although, we can obtain it from the general solution by letting $C \to \infty$.]

See the last part for a case when this "missing" solution is needed.

(b) Starting with $x(t) = \frac{1}{C - kt}$, we compute that $\frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{1}{(C - kt)^2} \cdot (-k) = \frac{k}{(C - kt)^2}$.

On the other hand, $kx^2 = k\left(\frac{1}{C - kt}\right)^2 = \frac{k}{(C - kt)^2}$. Since this matches what we got for $\frac{\mathrm{d}x}{\mathrm{d}t}$, it is indeed true that $\frac{\mathrm{d}x}{\mathrm{d}t} = kx^2$.

(c) We start with $x(t) = \frac{1}{C - kt}$ (which we know solves the DE for any value of C) and seek to choose C so that $x(0) = 2$.

Since $x(0) = \left[\frac{1}{C - kt}\right]_{t=0} = \frac{1}{C} \overset{!}{=} 2$, we find $C = \frac{1}{2}$.

Hence, the IVP has the (unique) solution $x(t) = \frac{1}{1/2 - kt}$.

(d) Proceeding as in the previous part, we now arrive at the impossible equation $\frac{1}{C} \overset{!}{=} 0$.

However, this suggests that we should consider taking $C \to \infty$ in $x(t) = \frac{1}{C - kt}$, which results in $x(t) = 0$.

Indeed, it is easy to verify (make sure you know what this entails!) that $x(t) = 0$ solves the IVP.

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