Notes for Lecture 1

Mon, 1/13/2025

Review: Computing derivatives

Given a function $y(x)$, we learned in Calculus I that its **derivative**

$$y'(x) = \frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

(where $\Delta y = y(x + \Delta x) - y(x)$) has the following two important characterizations:

- $y'(x)$ is the **slope of the tangent line** of the graph of $y(x)$ at x, and
- $y'(x)$ is the **rate of change** of $y(x)$ at x.

Comment. Derivatives were introduced in the late 1600s by Newton and Leibniz who later each claimed priority in laying the foundations for calculus. Certainly both of them contributed mightily to those foundations.

Moreover, we learned simple rules to compute the derivative of functions:

- **(sum rule)** $\frac{\mathrm{d}}{\mathrm{d}x}[f(x) + g(x)] = f'(x) + g'(x)$
- **(product rule)** $\frac{\mathrm{d}}{\mathrm{d}x}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$
- **(chain rule)** $\frac{\mathrm{d}}{\mathrm{d}x}[f(g(x))] = f'(g(x))g'(x)$

  Comment. If we write $t = g(x)$ and $y = f(t)$, then the chain rule takes the form $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t} \cdot \frac{\mathrm{d}t}{\mathrm{d}x}$.
  In other words, the chain rule expresses the fact that we can treat $\frac{\mathrm{d}y}{\mathrm{d}x}$ (which initially is just a notation for $y'(x)$) as an honest fraction.

- **(basic functions)** $\frac{\mathrm{d}}{\mathrm{d}x} x^r = r x^{r-1}$,
  $\frac{\mathrm{d}}{\mathrm{d}x} e^x = e^x$, $\frac{\mathrm{d}}{\mathrm{d}x} \ln(x) = \frac{1}{x}$,
  $\frac{\mathrm{d}}{\mathrm{d}x}\sin(x) = \cos(x)$, $\frac{\mathrm{d}}{\mathrm{d}x}\cos(x) = -\sin(x)$

These rules are enough to compute the derivative of any function that we can build from the basic functions using algebraic operations and composition. On the other hand, as you probably recall from Calculus II, reversing the operation of differentiation (i.e. computing antiderivatives) is much more difficult.

In particular, there exist simple functions (such as e^{x^2}) whose antiderivative cannot be expressed in terms of the basic functions above.

Example 1. Derive the **quotient rule** from the rules above.

Solution. We write $\frac{f(x)}{g(x)} = f(x) \cdot \frac{1}{g(x)}$ and apply the product rule to get

$$\frac{\mathrm{d}}{\mathrm{d}x} f(x) \cdot \frac{1}{g(x)} = f'(x)\frac{1}{g(x)} + f(x)\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{g(x)}.$$

By the chain rule combined with $\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{x} = -\frac{1}{x^2}$, we have $\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{g(x)} = -\frac{1}{g(x)^2} g'(x)$. Using this in the previous formula,

$$\frac{\mathrm{d}}{\mathrm{d}x} f(x) \cdot \frac{1}{g(x)} = f'(x)\frac{1}{g(x)} - f(x)\frac{1}{g(x)^2} g'(x) = \frac{f'(x)}{g(x)} - \frac{f(x)g'(x)}{g(x)^2}.$$

Putting the final two fractions on a common denominator, we obtain the familiar quotient rule

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{f(x)}{g(x)} = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$$

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Example 2. Compute the following derivatives:

(a) $\frac{\mathrm{d}}{\mathrm{d}x}(5x^3+7x^2+2)$

(b) $\frac{\mathrm{d}}{\mathrm{d}x}\sin(5x^3+7x^2+2)$

(c) $\frac{\mathrm{d}}{\mathrm{d}x}(x^3+2x)\sin(5x^3+7x^2+2)$

Solution.

(a) $\frac{\mathrm{d}}{\mathrm{d}x}(5x^3+7x^2+2)=15x^2+14x$

(b) $\frac{\mathrm{d}}{\mathrm{d}x}\sin(5x^3+7x^2+2)=(15x^2+14x)\cos(5x^3+7x^2+2)$

(c) $\frac{\mathrm{d}}{\mathrm{d}x}(x^3+2x)\sin(5x^3+7x^2+2)$
$=(3x^2+2)\sin(5x^3+7x^2+2)+(x^3+2x)(15x^2+14x)\cos(5x^3+7x^2+2)$

First examples of differential equations

Example 3. Here are two first examples of a **differential equation** (DE):

(a) $y'=2xy$

This is short for $y'(x)=2xy(x)$. The goal is to find a function $y(x)$ satisfying this equation.

One such **solution** is $y(x)=e^{x^2}$. We will soon learn techniques to find this ourselves but, already now, we can verify that it is indeed a solution: if $y(x)=e^{x^2}$ then $y'(x)=2xe^{x^2}=2xy(x)$.

(b) $(xy'-3y''')^2=5\sin(2x+y^4)+7$

This illustrates that y and its derivative can show up in any kind of way. We say that this DE has **order** 3 because the highest derivative is the 3rd derivative y'''.

Example 4. Solve the DE $y'=x^2+x$.

Solution. Note that the DE simply asks for a function $y(x)$ with a specific derivative (in particular, the right-hand side does not involve $y(x)$). In other words, the desired $y(x)$ is an **antiderivative** of x^2+x. We know from Calculus II that we can find antiderivatives by integrating:

$$y(x)=\int (x^2+x)\mathrm{d}x=\frac{1}{3}x^3+\frac{1}{2}x^2+C$$

Moreover, we know from Calculus II that there are no other solutions. In other words, we found the **general solution** to the DE.

Example 5. As in the previous example, any DE of the form $y'=f(x)$ (this is artificially easy) is just asking us to compute an antiderivative of $f(x)$.

On the other hand, this is an early indication that solving DEs is hard (and includes computing integrals as a special case). For instance, the DE $y'=e^{x^2}$ requires us to compute the antiderivative of e^{x^2}. It turns out that this cannot be done using the basic functions we know from Calculus.

Advanced comment. A "solution" to the above issue is to **define** a new function as the antiderivative that we presently cannot write down a formula for. Look up the so-called **error function** if you are curious!

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