Preparing Midterm #3

MATH 286 — Differential Equations Plus
April 17, 2014

The most exciting phrase to hear in science, the one that heralds new discoveries, is not "Eureka!" but "That's funny ...".
— Isaac Asimov (1920–1992) —

Problem 1. Let A be a 2×2 matrix such that $e^{At}=\begin{pmatrix} (1-t)e^{2t} & te^{2t} \\ -te^{2t} & (c+t)e^{rt} \end{pmatrix}$. What are the values of c and r?

Solution. Using that $e^{At}|_{t=0}=I$, we conclude that $c=1$. The presence of the term te^{2t} shows that 2 is a repeated eigenvalue. Since A is 2×2 and therefore has exactly 2 eigenvalues (counting with multiplicity), it follows that $r=2$ as well. □

Problem 2. Consider $\boldsymbol{x}'=A\boldsymbol{x}$ where $A=\begin{pmatrix} -2 & 1 \\ -6 & 3 \end{pmatrix}$.

(a) Find the general solution.

(b) Find e^{At}.

(c) Find a particular solution to $\boldsymbol{x}'=A\boldsymbol{x}+\begin{pmatrix} 1/t^2 \\ 2/t^2 \end{pmatrix}$.

Solution.

(a) The eigenvalues of A are $0,1$ with eigenvectors $\boldsymbol{v}=\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\boldsymbol{w}=\begin{pmatrix} 1 \\ 3 \end{pmatrix}$. Hence the general solution is

$$\boldsymbol{x}(t)=c_1\begin{pmatrix} 1 \\ 2 \end{pmatrix}+c_2\begin{pmatrix} 1 \\ 3 \end{pmatrix}e^t.$$

(b) From the first part, we know that a fundamental matrix is given by $\Phi(t)=\begin{pmatrix} 1 & e^t \\ 2 & 3e^t \end{pmatrix}$. Then

$$e^{At}=\Phi(t)\Phi(0)^{-1}=\begin{pmatrix} 1 & e^t \\ 2 & 3e^t \end{pmatrix}\begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix}=\begin{pmatrix} 3-2e^t & -1+e^t \\ 6-6e^t & -2+3e^t \end{pmatrix}.$$

(c) Using variation of constants,

$$\begin{aligned}
\boldsymbol{x}(t) &= e^{At}\int e^{-At}\begin{pmatrix} 1/t^2 \\ 2/t^2 \end{pmatrix}\mathrm{d}t \\
&= e^{At}\int \begin{pmatrix} 3-2e^{-t} & -1+e^{-t} \\ 6-6e^{-t} & -2+3e^{-t} \end{pmatrix}\begin{pmatrix} 1/t^2 \\ 2/t^2 \end{pmatrix}\mathrm{d}t \\
&= e^{At}\int \begin{pmatrix} 1/t^2 \\ 2/t^2 \end{pmatrix}\mathrm{d}t = e^{At}\begin{pmatrix} -1/t \\ -2/t \end{pmatrix} \\
&= \begin{pmatrix} 3-2e^t & -1+e^t \\ 6-6e^t & -2+3e^t \end{pmatrix}\begin{pmatrix} -1/t \\ -2/t \end{pmatrix}=\begin{pmatrix} -1/t \\ -2/t \end{pmatrix}.
\end{aligned}$$

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□

Problem 3. The mixtures in three tanks T_1, T_2, T_3 are kept uniform by stirring. Brine containing 2 lb of salt per gallon enters the first tank at 15 gal/min. Mixed solution from T_1 is pumped into T_2 at 10 gal/min and from T_2 into T_3 at 10 gal/min. Each tank initially contains 10 gal of pure water. Denote by $x_i(t)$ the amount (in pounds) of salt in tank T_i at time t (in minutes). Derive a system of linear differential equations for the x_i.

Solution. Note that at time t, T_1 contains $10+5t$ gal of solution. Likewise, T_2 contains 10 gal, and T_3 $10+10t$.

Consider a short interval of time $(t, t+\Delta t)$.

$$\begin{aligned}
\Delta x_1 &\approx 15\cdot 2\cdot \Delta t - 10\cdot \frac{x_1}{10+5t}\cdot \Delta t &&\Longrightarrow && x_1' = 30 - \frac{2x_1}{2+t}\\
\Delta x_2 &\approx 10\cdot \frac{x_1}{10+5t}\cdot \Delta t - 10\cdot \frac{x_2}{10}\cdot \Delta t &&\Longrightarrow && x_2' = \frac{2x_1}{2+t} - x_2\\
\Delta x_3 &\approx 10\cdot \frac{x_2}{10}\cdot \Delta t &&\Longrightarrow && x_3' = x_2
\end{aligned}$$

We also have the initial conditions $x_1(0)=0$, $x_2(0)=0$, $x_3(0)=0$. In matrix form, writing $\boldsymbol{x}=(x_1,x_2,x_3)$, this is

$$\boldsymbol{x}' = \begin{pmatrix} -\frac{2}{2+t} & 0 & 0\\ \frac{2}{2+t} & -1 & 0\\ 0 & 1 & 0 \end{pmatrix}\boldsymbol{x} + \begin{pmatrix} 30\\ 0\\ 0 \end{pmatrix}, \quad \boldsymbol{x}(0) = \begin{pmatrix} 0\\ 0\\ 0 \end{pmatrix}.$$

This is a system of linear inhomogeneous differential equations with non-constant coefficients (which means that we cannot apply our knowledge of eigenvectors to solve the complementary solution). □

Problem 4. Let A be a 3×3 matrix such that $e^{At} = \begin{pmatrix} e^{2t} - te^{-t} & te^{-t} & -e^{2t} + (t+1)e^{-t}\\ e^{2t} - e^{-t} & e^{-t} & -e^{2t} + e^{-t}\\ -te^{-t} & te^{-t} & (t+1)e^{-t} \end{pmatrix}$.

(a) What are the eigenvalues of A? Indicate if an eigenvalue is repeated and what its defect is.

(b) Solve the initial value problem $\boldsymbol{x}' = A\boldsymbol{x}$, $\quad \boldsymbol{x}(0) = (\ 1\ \ 0\ \ 1\ )^T$.

(c) Find a particular solution to $\boldsymbol{x}' = A\boldsymbol{x} + \begin{pmatrix} 1\\ 1\\ 0 \end{pmatrix}$.

(d) Find A. Also, as a challenge, find A^{100}.

Solution.

(a) The eigenvalues are $2, -1, -1$. The eigenvalue -1 is repeated and has defect 1.

(b) $\boldsymbol{x}(t) = e^{At}\begin{pmatrix} 1\\ 0\\ 1 \end{pmatrix} = \begin{pmatrix} e^{2t} - te^{-t}\\ e^{2t} - e^{-t}\\ -te^{-t} \end{pmatrix} + \begin{pmatrix} -e^{2t} + (t+1)e^{-t}\\ -e^{2t} + e^{-t}\\ (t+1)e^{-t} \end{pmatrix} = \begin{pmatrix} e^{-t}\\ 0\\ e^{-t} \end{pmatrix} = e^{-t}\begin{pmatrix} 1\\ 0\\ 1 \end{pmatrix}$.

(c) $\boldsymbol{x}_p(t) = e^{At}\int e^{-At}\begin{pmatrix} 1\\ 1\\ 0 \end{pmatrix}\mathrm{d}t = e^{At}\int \begin{pmatrix} e^{-2t}\\ e^{-2t}\\ 0 \end{pmatrix}\mathrm{d}t = -\frac{1}{2}e^{-2t}e^{At}\begin{pmatrix} 1\\ 1\\ 0 \end{pmatrix} = -\frac{1}{2}e^{-2t}\begin{pmatrix} e^{2t}\\ e^{2t}\\ 0 \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} 1\\ 1\\ 0 \end{pmatrix}$.

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(d) Recall that $\frac{\mathrm{d}}{\mathrm{d}t}e^{At} = Ae^{At}$. Setting $t=0$ then gives $A = \begin{pmatrix} 1 & 1 & -2 \\ 3 & -1 & -3 \\ -1 & 1 & 0 \end{pmatrix}$.

Our strategy to find A^{100} is to use $\frac{\mathrm{d}^{100}}{\mathrm{d}t^{100}}e^{At} = A^{100}\, e^{At}$. Clearly, $\frac{\mathrm{d}^{100}}{\mathrm{d}t^{100}}e^{2t} = 2^{100}e^{2t}$ and $\frac{\mathrm{d}^{100}}{\mathrm{d}t^{100}}e^{-t} = e^{-t}$. Note that $\frac{\mathrm{d}}{\mathrm{d}t}(te^{-t}) = (-t+1)e^{-t}$, $\frac{\mathrm{d}^2}{\mathrm{d}t^2}(te^{-t}) = (t-2)e^{-t}$, $\frac{\mathrm{d}^3}{\mathrm{d}t^3}(te^{-t}) = (-t+3)e^{-t}$. Continuing, we find that $\frac{\mathrm{d}^n}{\mathrm{d}t^n}(te^{-t}) = (-1)^n(t-n)e^{-t}$. In particular, $\frac{\mathrm{d}^{100}}{\mathrm{d}t^{100}}(te^{-t}) = (t-100)e^{-t}$. Therefore,

$$\frac{\mathrm{d}^{100}}{\mathrm{d}t^{100}}e^{At} = \begin{pmatrix} 2^{100}e^{2t} - (t-100)e^{-t} & (t-100)e^{-t} & -2^{100}e^{2t} + e^{-t} + (t-100)e^{-t} \\ 2^{100}e^{2t} - e^{-t} & e^{-t} & -2^{100}e^{2t} + e^{-t} \\ -(t-100)e^{-t} & (t-100)e^{-t} & e^{-t} + (t-100)e^{-t} \end{pmatrix}.$$

Setting $t=0$, we find

$$A^{100} = \begin{pmatrix} 2^{100} + 100 & -100 & -2^{100} - 99 \\ 2^{100} - 1 & 1 & -2^{100} + 1 \\ 100 & -100 & -99 \end{pmatrix}.$$

You should feel rightfully proud of your new powers! As you just discovered, the matrix exponential makes it possible to easily computer any power of a matrix (ours was a hard case because of the terms $te^{\lambda t}$ due to a defective eigenvalue; usually taking derivatives requires no thinking). □

Problem 5. The 6×6 matrix A has eigenvalues $-3, -3, 0, 1, 1, 1$.

(a) Which eigenvalues can be defective? Briefly describe in *all* possible scenarios what sort of (generalized) eigenvectors would arise, and what form the solutions take in each case.

(b) We wish to solve $\boldsymbol{x}' = A\boldsymbol{x} + \begin{pmatrix} 2t^2, & e^{-2t}\sin(t), & 0, & -1, & 0, & t\cos(t) \end{pmatrix}^T$. Write down a particular solution $\boldsymbol{x}_p$ with undetermined coefficients. It should have as few terms as possible and still work for any matrix A with the stated eigenvalues.

Solution.

(a) The eigenvalues $\lambda = -3$ and $\lambda = 1$ might be defective. $\lambda = -3$ may have no defect or defect 1. $\lambda = 1$ may have no defect or defect 1 or defect 2. We describe the possibilities individually.

$\boldsymbol{\lambda = -3}$ no defect. In this case, we find two (independent) eigenvectors for $\lambda = -3$.

$\boldsymbol{\lambda = -3}$ defect 1. There is one chain $\boldsymbol{v}_1, \boldsymbol{v}_2$ of generalized eigenvectors for $\lambda = -3$.

$\boldsymbol{\lambda = 1}$ no defect. Three (independent) eigenvectors for $\lambda = 1$.

$\boldsymbol{\lambda = 1}$ defect 1. There is one chain $\boldsymbol{v}_1$, $\boldsymbol{v}_2$ of generalized eigenvectors for $\lambda = 1$, as well as a second (independent) eigenvector $\boldsymbol{w}$ for $\lambda = 1$.

$\boldsymbol{\lambda = 1}$ defect 2. One chain $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ of generalized eigenvectors for $\lambda = 1$.

(b) There is a particular solution $\boldsymbol{x}_p$ of the form

$$\boldsymbol{x}_p(t) = (\boldsymbol{a}_1 t^3 + \boldsymbol{a}_2 t^2 + \boldsymbol{a}_3 t + \boldsymbol{a}_4) + \boldsymbol{b}_1 e^{-2t}\sin(t) + \boldsymbol{b}_2 e^{-2t}\cos(t) + (\boldsymbol{c}_1 t + \boldsymbol{c}_2)\cos(t) + (\boldsymbol{d}_1 t + \boldsymbol{d}_2)\sin(t).$$

Note that this is $10 \times 6 = 60$ undeterminates. Let's keep our theoretical attitude... □

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Problem 6. Consider $\boldsymbol{x}' = A\boldsymbol{x}$ where $A = \begin{pmatrix} 2 & 4 & -1 \\ 7 & -1 & -5 \\ -1 & 1 & -1 \end{pmatrix}$.

(a) Find a fundamental matrix. *Hint:* The eigenvalues of A are $-3, -3, 6$.

(b) Solve the initial value problem with $\boldsymbol{x}(0) = \begin{pmatrix} 3 & 0 & 0 \end{pmatrix}^T$.

Solution.

(a) The eigenvalues of A are $-3, -3, 6$.

For $\lambda = 6$ we find the eigenvector $\boldsymbol{v} = \begin{pmatrix} 1 & 1 & 0 \end{pmatrix}^T$.

For $\lambda = -3$ we find the eigenvector $\boldsymbol{w}_1 = \begin{pmatrix} 1 & -1 & 1 \end{pmatrix}^T$ but no second one. $\lambda = -3$ thus has defect 1.

Therefore there has to be a chain of length 2.

We solve $\begin{pmatrix} 5 & 4 & -1 \\ 7 & 2 & -5 \\ -1 & 1 & 2 \end{pmatrix} \boldsymbol{w}_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ and find, for instance, $\boldsymbol{w}_2 = \begin{pmatrix} 0 \\ 1/3 \\ 1/3 \end{pmatrix}$.

Fundamental matrix: $\Phi(t) = \begin{pmatrix} e^{6t} & e^{-3t} & te^{-3t} \\ e^{6t} & -e^{-3t} & (1/3 - t)e^{-3t} \\ 0 & e^{-3t} & (1/3 + t)e^{-3t} \end{pmatrix}$.

(b) We need to find $\boldsymbol{c}$ such that $\boldsymbol{x}(t) = \Phi(t)\boldsymbol{c}$ satisfies $\boldsymbol{x}(0) = \begin{pmatrix} 3 & 0 & 0 \end{pmatrix}^T$.

$\boldsymbol{x}(0) = \Phi(0)\boldsymbol{c} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 1/3 \\ 0 & 1 & 1/3 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \overset{!}{=} \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$. We solve and find $\boldsymbol{c} = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$.

Hence $\boldsymbol{x}(t) = \begin{pmatrix} 2e^{6t} + (1 - 3t)e^{-3t} \\ 2e^{6t} - (2 - 3t)e^{-3t} \\ -3te^{-3t} \end{pmatrix}$. □

Problem 7. Let $A = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & 0 \\ 0 & 1 & 2 \end{pmatrix}$.

(a) Show that the matrix $N = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ is nilpotent.

(b) Use the fact that N is nilpotent, to find e^{At}.

(c) Solve the initial value problem $\boldsymbol{x}' = A\boldsymbol{x}$, $\quad \boldsymbol{x}(0) = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}^T$.

(d) Find a particular solution of $\boldsymbol{x}' = A\boldsymbol{x} + \begin{pmatrix} e^{2t} \\ -e^t \\ 0 \end{pmatrix}$.

(e) Use a different method to solve the previous problem.

Solution.

(a) We compute $N^2 = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $N^3 = 0$.

(b) Note that $At = 2It + Nt$. Since the matrices $2It$ and Nt commute (why?!), we have

$$\begin{aligned} e^{At} = e^{2It} e^{Nt} &= \begin{pmatrix} e^{2t} & & \\ & e^{2t} & \\ & & e^{2t} \end{pmatrix} \left[I + Nt + \frac{1}{2}(Nt)^2 \right] \\ &= e^{2t} \left[I + \begin{pmatrix} 0 & 0 & -t \\ 0 & 0 & 0 \\ 0 & t & 0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 0 & -t^2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right] = \begin{pmatrix} 1 & -t^2/2 & -t \\ 0 & 1 & 0 \\ 0 & t & 1 \end{pmatrix} e^{2t}. \end{aligned}$$

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(c) This is easy now! $\boldsymbol{x}(t) = e^{At}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1-3t-t^2 \\ 2 \\ 3+2t \end{pmatrix} e^{2t}$.

(d) Using variation of constants, a particular solution is

$$\boldsymbol{x}_p(t) = e^{At}\int e^{-At}\begin{pmatrix} e^{2t} \\ -e^t \\ 0 \end{pmatrix} \mathrm{d}t = e^{At}\int \begin{pmatrix} 1 & -t^2/2 & t \\ 0 & 1 & 0 \\ 0 & -t & 1 \end{pmatrix} e^{-2t}\begin{pmatrix} e^{2t} \\ -e^t \\ 0 \end{pmatrix} \mathrm{d}t = e^{At}\int \begin{pmatrix} 1+t^2/2e^{-t} \\ -e^{-t} \\ te^{-t} \end{pmatrix} \mathrm{d}t$$

$$= \begin{pmatrix} 1 & -t^2/2 & -t \\ 0 & 1 & 0 \\ 0 & t & 1 \end{pmatrix} e^{2t}\begin{pmatrix} t - t^2/2e^{-t} - te^{-t} - e^{-t} \\ e^{-t} \\ -te^{-t} - e^{-t} \end{pmatrix} = \begin{pmatrix} te^{2t} - e^t \\ e^t \\ -e^t \end{pmatrix}.$$

(e) Let us use undetermined coefficients. From e^{At} we know that A has eigenvalues $2, 2, 2$. Let us split the problem into two parts: $\boldsymbol{x}' = A\boldsymbol{x} + e^t(0, -1, 0)^T$ and $\boldsymbol{x}' = A\boldsymbol{x} + e^{2t}(1, 0, 0)^T$.

For the first part, we look for a solution of the form $\boldsymbol{x}_p = \boldsymbol{a}e^t$. Plugging into the DE, we get

$$\boldsymbol{x}' = \boldsymbol{a}e^t \overset{!}{=} A\boldsymbol{x} + \boldsymbol{f} = A\boldsymbol{a}e^t + e^t\begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} \iff (A - I)\boldsymbol{a} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}\boldsymbol{a} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \iff \boldsymbol{a} = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}.$$

In the other case, severe duplication occurs. We know that, in the worst case, there is a solution $\boldsymbol{x}_p$ of the form $\boldsymbol{x}_p = (\boldsymbol{a} + \boldsymbol{b}t + \boldsymbol{c}t^2 + \boldsymbol{d}t^3)e^{2t}$. For practical purposes, and when working by hand, it still makes sense to first look if there exists a simpler solution $\boldsymbol{x}_p = \boldsymbol{a}e^{2t}$ before adding a power of t each time we find no solution.

$$\boldsymbol{x}' = 2\boldsymbol{a}e^{2t} \overset{!}{=} A\boldsymbol{x} + \boldsymbol{f} = A\boldsymbol{a}e^{2t} + e^{2t}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \iff (A - 2I)\boldsymbol{a} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\boldsymbol{a} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \iff \boldsymbol{a} = \begin{pmatrix} c \\ 0 \\ 1 \end{pmatrix}.$$

We are lucky and have already found a solution (we can set c to anything, for instance $c = 0$)!

Combining, we have the particular solution (of the original problem) $\boldsymbol{x}_p = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} e^t + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{2t} = \begin{pmatrix} -e^t \\ e^t \\ e^{2t} - e^t \end{pmatrix}$.

Note that this looks rather different from the solution found in the previous problem. This is explained by

$$\begin{pmatrix} -e^t \\ e^t \\ e^{2t} - e^t \end{pmatrix} = \begin{pmatrix} te^{2t} - e^t \\ e^t \\ -e^t \end{pmatrix} + \begin{pmatrix} -te^{2t} \\ 0 \\ e^{2t} \end{pmatrix}, \quad \text{with } \begin{pmatrix} -te^{2t} \\ 0 \\ e^{2t} \end{pmatrix} = e^{At}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \text{ a solution of } \boldsymbol{x}' = A\boldsymbol{x}.$$

□

Problem 8. Find the general solution of

$$\boldsymbol{x}' = \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}\boldsymbol{x}.$$

You may use that the characteristic polynomial has the repeated roots $1 \pm i$. The general solution should be given in terms of real-valued functions.

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Solution. Let us find the eigenvectors for the eigenvalue $\lambda = 1 - i$.

$$\left[\begin{array}{cccc|c} i & 1 & -1 & 0 & 0 \\ 0 & i & 0 & -1 & 0 \\ 1 & 0 & i & 1 & 0 \\ 0 & 1 & 0 & i & 0 \end{array}\right] \overset{r_3 = r_3 + i r_1}{\Longrightarrow} \left[\begin{array}{cccc|c} i & 1 & -1 & 0 & 0 \\ 0 & i & 0 & -1 & 0 \\ 0 & i & 0 & 1 & 0 \\ 0 & 1 & 0 & i & 0 \end{array}\right] \overset{\substack{r_3 = r_3 - r_2 \\ r_4 = r_4 + i r_2}}{\Longrightarrow} \left[\begin{array}{cccc|c} i & 1 & -1 & 0 & 0 \\ 0 & i & 0 & -1 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \Longrightarrow \boldsymbol{v} = \begin{pmatrix} c \\ 0 \\ i c \\ 0 \end{pmatrix}$$

Let us choose $c = 1$. There was only one degree of freedom, so the defect of λ is 1. We have to construct a chain starting with $\boldsymbol{v}_1 = (1, 0, i, 0)^T$. We extend the above elimination:

$$\left[\begin{array}{cccc|cc} i & 1 & -1 & 0 & 0 & 1 \\ 0 & i & 0 & -1 & 0 & 0 \\ 1 & 0 & i & 1 & 0 & i \\ 0 & 1 & 0 & i & 0 & 0 \end{array}\right] \overset{r_3 = r_3 + i r_1}{\Longrightarrow} \left[\begin{array}{cccc|cc} i & 1 & -1 & 0 & 0 & 1 \\ 0 & i & 0 & -1 & 0 & 0 \\ 0 & i & 0 & 1 & 0 & 2i \\ 0 & 1 & 0 & i & 0 & 0 \end{array}\right] \overset{\substack{r_3 = r_3 - r_2 \\ r_4 = r_4 + i r_2}}{\Longrightarrow} \left[\begin{array}{cccc|cc} i & 1 & -1 & 0 & 0 & 1 \\ 0 & i & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 2i \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \Longrightarrow \boldsymbol{v}_2 = \begin{pmatrix} c \\ 1 \\ i c \\ i \end{pmatrix}$$

We again choose $c = 0$. Summarizingly, we found the two complex solutions

$$\boldsymbol{x}_1 = \begin{pmatrix} 1 \\ 0 \\ i \\ 0 \end{pmatrix} e^{(1-i)t}, \quad \boldsymbol{x}_2 = \left[\begin{pmatrix} 1 \\ 0 \\ i \\ 0 \end{pmatrix} t + \begin{pmatrix} 0 \\ 1 \\ 0 \\ i \end{pmatrix} \right] e^{(1-i)t} = \begin{pmatrix} t \\ 1 \\ i t \\ i \end{pmatrix} e^{(1-i)t}.$$

(Together with their conjugates, we actually have four independent solutions). By taking real and imaginary parts (recall that $e^{(1-i)t} = e^t(\cos(t) - i \sin(t))$), we conclude that four independent real solutions are given by

$$\operatorname{Re}(\boldsymbol{x}_1) = e^t \begin{pmatrix} \cos(t) \\ 0 \\ \sin(t) \\ 0 \end{pmatrix}, \quad \operatorname{Im}(\boldsymbol{x}_1) = e^t \begin{pmatrix} -\sin(t) \\ 0 \\ \cos(t) \\ 0 \end{pmatrix}, \quad \operatorname{Re}(\boldsymbol{x}_2) = e^t \begin{pmatrix} t \cos(t) \\ \cos(t) \\ t \sin(t) \\ \sin(t) \end{pmatrix}, \quad \operatorname{Im}(\boldsymbol{x}_2) = e^t \begin{pmatrix} -t \sin(t) \\ -\sin(t) \\ t \cos(t) \\ \cos(t) \end{pmatrix}.$$

□

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