Preparing Midterm #1

MATH 286 — Differential Equations Plus
February 13, 2014

Problem 1. Without solving the equation $(y-2)y' = x + e^y + 1$, answer the following questions.

(a) Does the existence/uniqueness theorem guarantee the existence of a solution to the above equation with initial condition $y(2) = 0$? If so, does it guarantee the solution to be unique?

(b) Same question for the initial condition $y(0) = 2$.

(c) Sketch the slope field of the differential equation. What does it suggest regarding the previous questions?

(d) Consider the solution with initial condition $y(1) = 0$. Find the equation for its tangent line at the point $(1, 0)$.

(e) Again, considering the solution with initial condition $y(1) = 0$, what is $y''(1)$?

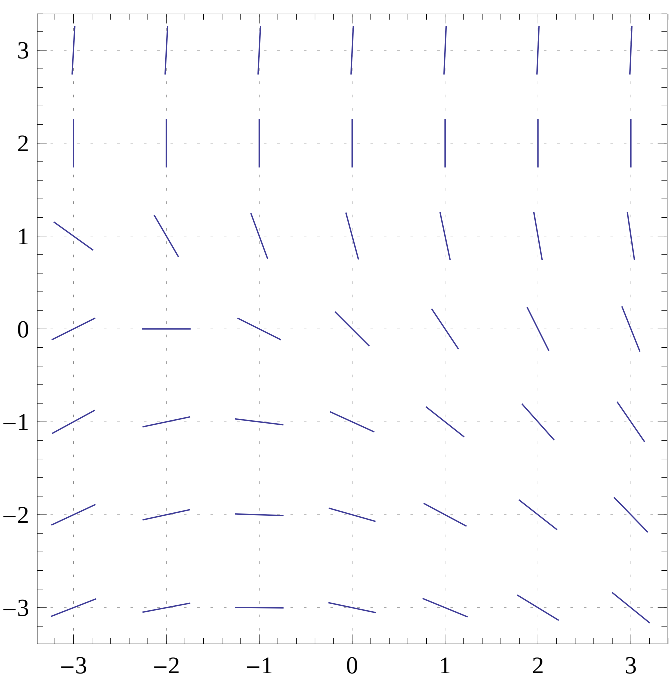
Solution. Here, $y' = f(x, y)$ with $f(x, y) = \frac{x + e^y + 1}{y - 2}$.

(a) The function $f(x, y)$ is continuous at all points (x, y) for which $y \neq 2$. This includes the point $(2, 0)$ and a neighborhood of it. Hence there exists a solution with initial condition $y(2) = 0$.

$\frac{\partial}{\partial y} f(x, y) = \frac{e^y(y-2) - (x + e^y + 1)}{(y-2)^2}$ is again continuous at all points (x, y) for which $y \neq 2$. Hence the solution with initial condition $y(2) = 0$ is unique.

(b) The function $f(x, y)$ is not continuous at the point $(0, 2)$. Hence the existence/uniqueness theorem does not guarantee the existence of a solution with initial condition $y(0) = 2$.

(c) (If we look very carefully; sorry for that) the slope field below suggests that, if we permit the slope $y'(0)$ to be infinite, there should be two solutions to the DE with initial condition $y(0) = 2$ (very much like in the example $y' = -x/y$, $y(a) = 0$, from class where we obtained half-circles as solution functions). (If you are a strict person, it would also be OK to say that no solution exists because $y'(0)$ would be undefined, whereas a solution to a differential equation should be differentiable. The important part is to be aware of the issues.)



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(d) Plugging $y(1)=0$ into the equation yields $-2y'(1)=1+1+1$. Hence $y'(1)=-\frac{3}{2}$ and the tangent line at $(1,0)$ has equation $y=-\frac{3}{2}(x-1)$.

(e) Apply $\frac{\mathrm{d}}{\mathrm{d}x}$ to the DE to get $y'y'+(y-2)y''=1+e^y y'$. Using $y(1)=0$ and $y'(1)=-\frac{3}{2}$, we get $\left(-\frac{3}{2}\right)^2-2y''(1)=1-\frac{3}{2}$ and hence $y''(1)=\frac{11}{8}$. □

Problem 2. Solve the initial value problem $y'=2xy+3x^2e^{x^2}$, $y(0)=5$.

Solution. This equation is linear: $y'-2xy=3x^2e^{x^2}$

The integrating factor is $e^{\int -2x\,\mathrm{d}x}=e^{-x^2}$:

$$\begin{aligned} e^{-x^2}y'-e^{-x^2}2xy &= 3x^2 \\ \frac{\mathrm{d}}{\mathrm{d}x}\left[e^{-x^2}y\right] &= \frac{\mathrm{d}}{\mathrm{d}x}[x^3] \\ e^{-x^2}y &= x^3+C \end{aligned}$$

Using $y(0)=5$, we find $5=C$. Hence the solution is $y=(x^3+5)e^{x^2}$. □

Problem 3. Find a general solution of the equation $x(x+y)y'=y(3x+y)$.

Solution. This equation is homogeneous as can be seen by dividing by xy to get $\left(\frac{x}{y}+1\right)y'=3+\frac{y}{x}$.

Substituting $u=\frac{y}{x}$ gives $\frac{\mathrm{d}y}{\mathrm{d}x}=u+x\frac{\mathrm{d}u}{\mathrm{d}x}$ and hence:

$$\begin{aligned} (u^{-1}+1)(u+xu') &= 3+u \\ (1+u^{-1})u' &= \frac{2}{x} \\ u+\ln|u| &= 2\ln|x|+C \\ |u|\,e^u &= e^C x^2 \end{aligned}$$

Assuming $u>0$, we get $ue^u=Dx^2$ with $D>0$. This implicit solution cannot be made more explicit using standard functions. For the original problem, we get the implicit solution $ye^{y/x}=Dx^3$. □

Problem 4. Find a general solution of the equation $2+\frac{\mathrm{d}y}{\mathrm{d}x}=\sqrt{2x+y}$.

Solution. Substitute $u=2x+y$. Then $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}u}{\mathrm{d}x}-2$ and we get $\frac{\mathrm{d}u}{\mathrm{d}x}=\sqrt{u}$ which is separable.

$2u^{1/2}=x+C$, hence $u=\frac{1}{4}(x+C)^2$ and $y=u-2x=\frac{1}{4}(x+C)^2-2x$ [which is a solution as long as $x+C>0$]. □

Problem 5. In a city with a fixed population P, the time rate of change of the number N of people who have heard a certain rumor is proportional to N and $P-N$. Suppose initially 10% have heard the rumor and after a week this number has grown to 20%. What percentage will this number reach after one more week?

Solution. $\frac{\mathrm{d}N}{\mathrm{d}t}=kN(P-N)$. $N(0)=0.1P$ and $N(1)=0.2P$. We need $N(2)$.

$$\int\frac{\mathrm{d}N}{N(P-N)}=\frac{1}{P}\int\left[\frac{1}{N}+\frac{1}{P-N}\right]\mathrm{d}N=\int k\,\mathrm{d}t$$

Hence $\frac{1}{P}\ln\left|\frac{N}{P-N}\right|=kt+C$. Using $N(0)=0.1P$ and $N(1)=0.2P$, we have $\frac{1}{P}\ln\frac{1}{9}=C$ and $\frac{1}{P}\ln\frac{1}{4}=k+C$.

This gives $\ln\left|\frac{N}{P-N}\right|=\left(\ln\frac{9}{4}\right)t+\ln\frac{1}{9}$ and hence $\frac{N}{P-N}=\frac{1}{9}\left(\frac{9}{4}\right)^t$.

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When $t=2$ thus $\dfrac{N(2)}{P-N(2)}=\dfrac{9}{16}$ and, solving for $N(2)$, we find $\frac{25}{16}N(2)=\frac{9}{16}P$, thus $N(2)=\frac{9}{25}P$ which is 36%. □

Problem 6. Solve the initial value problem $y''-5y'+6y=0,\ \ y(0)=0, y'(0)=1$.

Solution. This is a linear homogeneous equation with constant coefficients. The characteristic equation is $r^2-5r+6=(r-2)(r-3)$. Hence the general solution is $y(x)=c_1\, e^{2x}+c_2\, e^{3x}$. Solving

$$\begin{aligned} y(0)=0 &= c_1+c_2 \\ y'(0)=1 &= 2c_1+3c_2 \end{aligned}$$

gives $c_1=-1$ and $c_2=1$. The solution is $y(x)=e^{3x}-e^{2x}$. □

Problem 7. Solve: $\quad x^2\dfrac{\mathrm{d}y}{\mathrm{d}x}=x y-x^2 e^{y/x}, \qquad y(1)=0$

Solution. $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{y}{x}-e^{y/x}$ is homogeneous. Substitute $u=\frac{y}{x}$ to get $u+x\frac{\mathrm{d}u}{\mathrm{d}x}=u-e^u$ or $x\frac{\mathrm{d}u}{\mathrm{d}x}=-e^u$.

By separation of variables $e^{-u}=\ln|x|+C$. Using $u(1)=\frac{0}{1}=0$ we find $C=1$.

Hence $-u=\ln(1+\ln x)$ and therefore $y=u x=-x\ln(1+\ln x)$. □

Problem 8. Find a general solution of the equation $x y'=y+x^2\cos(x)$.

Solution. This equation is linear: $y'-\frac{1}{x}\, y=x\cos(x)$

The integrating factor is $e^{\int -\frac{1}{x}\mathrm{d}x}=e^{-\ln x}=\frac{1}{x}$:

$$\begin{aligned} \frac{1}{x}\, y'-\frac{1}{x^2}\, y &= \cos(x) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{x}\, y\right] &= \frac{\mathrm{d}}{\mathrm{d}x}[\sin(x)] \\ \frac{y}{x} &= \sin(x)+C \end{aligned}$$

Hence a general solution is $y=x\sin(x)+Cx$. □

Problem 9. Find the general solution to $y^{(5)}-4y^{(4)}+5y'''-2y''=0$.

Solution. This is a linear homogeneous equation with constant coefficients. The characteristic equation is $r^5-4r^4+5r^3-2r^2=r^2(r-1)^2(r-2)$. Hence the general solution is

$$y(x)=c_1+c_2x+(c_3+c_4x)e^x+c_5e^{2x}.$$

□

Problem 10. Write down a homogeneous linear differential equation satisfied by $y(x)=1-5x^2e^{-2x}$.

Solution. A linear homogeneous DE with constant coefficients will do if its characteristic equation is $r(r+2)^3=0$. Since $r(r+2)^3=r^4+6r^3+12r^2+8r$, a corresponding DE is

$$y^{(4)}+6y'''+12y''+8y'=0.$$

□

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