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MATH 286 SECTION X1 – Introduction to Differential Equations Plus

MIDTERM EXAMINATION 1

September 25, 2013

INSTRUCTOR: M. BRANNAN

INSTRUCTIONS

- This exam 60 minutes long. No personal aids or calculators are permitted.
- Answer all questions in the space provided. If you require more space to write your answer, you may continue on the back of the page. There is a blank page at the end of the exam for rough work.
- EXPLAIN YOUR WORK! Little or no points will be given for a correct answer with no explanation of how you got it. If you use a theorem to answer a question, indicate which theorem you are using, and explain why the hypotheses of the theorem are valid.
- GOOD LUCK!

PLEASE NOTE: "Proctors are unable to respond to queries about the interpretation of exam questions. Do your best to answer exam questions as written."

Question:	1	2	3	4	5	Total
Points:	6	9	10	8	7	40
Score:						

1. (6 points) Solve the initial value problem

$$\frac{dy}{dx} = y^{-1} e^{y} \cos x, \qquad y(0) = 1.$$

An implicit expression for the solution is fine.

Solution: This is a separable equation:

$$ye^{-y}dy = \cos x dx \implies \int ye^{-y}dy = \int \cos x dx + C \implies -ye^{-y} - e^{-y} = \sin x + C.$$

To find C, we plug in $y(0) = 1$, which gives $-2e^{-1} = C$. The solution is therefore

$$ye^{-y} + e^{-y} = 2e^{-1} - \sin x.$$

2. Consider the ordinary differential equation

$$\frac{dy}{dx} = \frac{y}{x} + x \ln x \qquad (x > 0).$$

(a) (3 points) Without explicitly solving this ODE, determine whether the corresponding initial value problem with initial condition $y(1) = 0$ has a unique solution, no solution, or more than one solution. *Explain your answer!*

Solution: Let $F(x, y) = \frac{y}{x} + x \ln x$. According to the existence-uniqueness theorem for initial value problems, there exists a unique solution to the above IVP if F and $\frac{\partial F}{\partial y} = \frac{1}{x}$ are both continuous on a rectangle containing the initial data $(1, y(1)) = (1, 0)$. Since this is indeed the case, there is a unique solution.

(b) (6 points) Find the solution to the IVP

$$\frac{dy}{dx} = \frac{y}{x} + x \ln x, \qquad y(1) = 1.$$

Solution: We can write this equation as $\frac{dy}{dx} - \frac{y}{x} = x \ln x$, which is obviously linear, with integrating factor

$$e^{P(x)} = e^{\int -x^{-1} dx} = e^{-\ln x} = \frac{1}{x}.$$

Multiplying through by this integrating factor, we get

$$\begin{aligned} \frac{1}{x}\frac{dy}{dx} - \frac{1}{x^2} y = \ln x &\iff \frac{d}{dx}\left(\frac{y}{x}\right) = \ln x \\ \implies \frac{y}{x} &= \int \ln x dx + C = x \ln x - x + C \\ &\implies y = x^2 \ln x - x^2 + Cx. \end{aligned}$$

Plugging in the initial conditions we get $C = 2$.

3. Consider the ODE

$$\frac{x}{x^2+y^2+1} - \sin x + \left(\frac{y}{x^2+y^2+1}\right)\frac{dy}{dx} = 0 \qquad ((x,y) \neq (0,0)).$$

(a) (4 points) Show that this ODE is exact.

Solution: Let $M(x,y) = \frac{x}{x^2+y^2+1} - \sin x$ and $N(x,y) = \frac{y}{x^2+y^2+1}$. Then

$$M_y = \frac{-2xy}{(x^2+y^2+1)^2} \quad \& \quad N_x = \frac{-2xy}{(x^2+y^2+1)^2}.$$

Since $N_x = M_y$, the equation is exact.

(b) (6 points) Find an implicit expression for the general solution to this ODE.

Solution: We need to find a function $F(x,y)$, defined for $(x,y) \neq (0,0)$, such that our solution curves lie along the level sets $F(x,y) = C$. If this is the case, then we must have $M = F_x$ and $N = F_y$. Solving these equations, we obtain

$$\begin{aligned} F(x,y) &= \int M(x,y)dx = \int \left(\frac{x}{x^2+y^2+1} - \sin x\right)dx \\ &= \frac{1}{2}\ln(x^2+y^2+1) + \cos x + g(y), \end{aligned}$$

where $g(y)$ is some unknown function of y. To find $g(y)$, we use the equation $F_y = N$, which gives

$$N = \frac{y}{x^2+y^2+1} = F_y = \frac{y}{x^2+y^2+1} + g'(y) \implies g'(y) = 0.$$

Therefore $g(y) = K$ is constant and our solutions are on the level curves

$$F(x,y) = \frac{1}{2}\ln(x^2+y^2+1) + \cos x + K = C.$$

4. (8 points) Find the general solution to the second order ODE

$$xy'' = x\exp\left(\frac{y'}{x}\right) + y' \qquad (x > 0),$$

where $\exp a = e^a$. (**Note:** It is OK if your final answer involves an indefinite integral that cannot be evaluated).

Solution: Making the substitution $z = y'$, the equation becomes first order:

$$xz' = x\exp\left(\frac{z}{x}\right) + z \iff z' = \exp\left(\frac{z}{x}\right) + \frac{z}{x}.$$

This is now a homogeneous substitution problem. Set $v = \frac{z}{x}$. Then $z' = xv' + v$, which gives

$$xv' + v = e^v + v \implies e^{-v}dv = \frac{dx}{x} \implies -e^{-v} = \ln x + C.$$

Solving for v, we get

$$\frac{z}{x} = v = -\ln(-\ln x - C) \implies z = -x\ln(-\ln x - C)$$

To get y, we integrate:

$$y(x) = \int z(x)dx + K = \int -x\ln(-\ln x + C)dx + K.$$

5. (7 points) The functions

$$y_1(x) = e^x, \quad y_2(x) = e^{-x}, \quad y_3(x) = e^{2x} \qquad (x \in \mathbb{R}),$$

are solutions to the 3rd order linear ODE

$$y''' + Ay'' + By' + Cy = 0.$$

What are A, B, C?

Solution: The characteristic polynomial for this ODE is $P(r) = r^3 + Ar^2 + Br + C$. On the other hand, since y_1, y_2, y_3 are solutions, we know that $P(r)$ has roots $1, -1$, and 2 (each with multiplicity 1). Thus

$$P(r) = (r-1)(r+1)(r-2) = (r^2-1)(r-2) = r^3 - 2r^2 - r + 2.$$

Comparing coefficients, we obtain

$$A = -2, \; B = -1, \; C = 2.$$

(*Extra work space.*)