Review. generalized eigenvectors and corresponding solutions to $\boldsymbol{x}' = A\boldsymbol{x}$

Recipe for solving $\boldsymbol{x}' = A\boldsymbol{x}$:

- find eigenvalues λ
- for each λ, find eigenvectors
- if λ is defective, find enough chains[21]
- if $\lambda = a \pm bi$ is complex, take real and imaginary part of the solutions found ◇

Example 127. Find the general solution of $\boldsymbol{x}' = \begin{pmatrix} 0 & 1 & 2 \\ -5 & -3 & -7 \\ 1 & 0 & 0 \end{pmatrix}\boldsymbol{x}$.

Solution. The characteristic polynomial is ... $= -(\lambda+1)^3$.

Hence, $\lambda = -1$ is an eigenvalue of multiplicity 3.

We first solve for eigenvectors: $\left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ -5 & -2 & -7 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right] \underset{\substack{r_2+5r_1 \\ r_3-r_1}}{\Longrightarrow} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & -1 & -1 & 0 \end{array}\right] \underset{\substack{r_2/3 \\ 3r_3+r_2}}{\Longrightarrow} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

Setting $v_3 = c$, we get $v_2 = -c$ and then $v_1 = -c$. The choice $c = -1$ gives $\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$. The corresponding solution is $\boldsymbol{x}_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} e^{-t}$.

Since there was only one degree of freedom, there is no other independent eigenvector. $\lambda = -1$ has defect 2.

Because there is only one eigenvector to build a chain upon, we now know that there has to be a chain $\boldsymbol{v}_1, \boldsymbol{v}_2$, $\boldsymbol{v}_3$ of three generalized eigenvectors.

To find $\boldsymbol{v}_2$, we need to solve $\begin{pmatrix} 1 & 1 & 2 \\ -5 & -2 & -7 \\ 1 & 0 & 1 \end{pmatrix}\boldsymbol{v}_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$. We can save time and effort by reusing the elimination we have already done: $\left[\begin{array}{ccc|cc} 1 & 1 & 2 & 0 & 1 \\ -5 & -2 & -7 & 0 & 1 \\ 1 & 0 & 1 & 0 & -1 \end{array}\right] \underset{\substack{r_2+5r_1 \\ r_3-r_1}}{\Longrightarrow} \left[\begin{array}{ccc|cc} 1 & 1 & 2 & 0 & 1 \\ 0 & 3 & 3 & 0 & 6 \\ 0 & -1 & -1 & 0 & -2 \end{array}\right] \underset{\substack{r_2/3 \\ 3r_3+r_2}}{\Longrightarrow} \left[\begin{array}{ccc|cc} 1 & 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

This time, $v_3 = c$ leads to $v_2 = 2 - c$. $v_1 + v_2 + 2v_3 = 1$ then gives $v_1 = -1 - c$. Hence, $\boldsymbol{v}_2 = \begin{pmatrix} -1-c \\ 2-c \\ c \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} + c\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$. Note that the second summand is just an eigenvector! We can choose any c. For instance, choosing $c = 0$ gives $\boldsymbol{v}_2 = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$ with corresponding solution[22] $\boldsymbol{x}_2 = \left[\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} t + \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}\right] e^{-t}$.

Finally, to find $\boldsymbol{v}_3$ we have to solve $(A - \lambda I)\boldsymbol{v}_3 = \boldsymbol{v}_2$. We can again reuse the elimination we have already done:

$\left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 0 & 1 & -1 \\ -5 & -2 & -7 & 0 & 1 & 2 \\ 1 & 0 & 1 & 0 & -1 & 0 \end{array}\right] \underset{\substack{r_2+5r_1 \\ r_3-r_1}}{\Longrightarrow} \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 0 & 1 & -1 \\ 0 & 3 & 3 & 0 & 6 & -3 \\ 0 & -1 & -1 & 0 & -2 & 1 \end{array}\right] \underset{\substack{r_2/3 \\ 3r_3+r_2}}{\Longrightarrow} \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 0 & 1 & -1 \\ 0 & 1 & 1 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$

As before, $v_3 = c$ leads to $v_2 = -1 - c$. $v_1 + v_2 + 2v_3 = -1$ then gives $v_1 = -c$. Choosing $c = 0$, we get $\boldsymbol{v}_3 = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$ which gives us the solution[23] $\boldsymbol{x}_3 = \left[\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \frac{t^2}{2} + \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} t + \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}\right] e^{-t}$. ◇

Example 128. Suppose we have an eigenvalue λ of multiplicity 5.

Here are the 7 possibilities for chains, listed by the lengths of the chains that occur:

- (defect 0) $1, 1, 1, 1, 1$ [i.e., 5 eigenvectors]
- (defect 1) $2, 1, 1, 1$
- (defect 2) $2, 2, 1$ or $3, 1, 1$
- (defect 3) $3, 2$ or $4, 1$
- (defect 4) 5

Note that the defect is something we know (after computing the eigenvectors). We have seen how to do the defect 0 and defect 4 cases; the other ones are a little bit more intricate. ◇

21. These computations can become a bit intricate. For exams, we will content ourselves with the defective cases involving a single chain (per eigenvalue) only.

22. How does choosing a different c affect the solution $\boldsymbol{x}_2$. Why does it not make a difference?

23. Try and see what happens if you went looking for a fourth vector $\boldsymbol{v}_4$ in the chain. Why does it fail?

Armin Straub
astraub@illinois.edu