Remark 122. One of the problems on the midterm asked for solving $y''' - y = e^x + 7$.

The characteristic polynomial $r^3 - 1$ has root $r = 1$. To find the other roots, we can do polynomial division to get $r^3 - 1 = (r-1)(r^2 + r + 1)$.
More generally, the roots of $z^n - 1$ are called n-th roots of unity. $z^n = 1$ implies that $|z| = 1$, which means that these numbers lie on the unit circle. In particular, they are of the form $e^{i\theta}$. Remembering that $e^{2\pi i} = 1$, we find that $\zeta = e^{2\pi i/n}$ is a n-th root of unity and so are $\zeta^2 = e^{4\pi i/n}$, $\zeta^3 = e^{6\pi i/n}$, ...
Geometrically, the n-th roots of unity form the vertices of a regular n-gon. Now, go back to the equation $z^3 - 1$ and mark the solutions on the unit circle. ◇

Remark 123. Another problem on the midterm asked to find a homogeneous linear DE solved by solutions of the inhomogeneous linear DE $y'' + xy = e^x$.

Note that this DE does not have constant coefficients. Yet, we can proceed as we did in the case of constant coefficients: e^x solves a HLDE with constant coefficients and root 1 (the "new" root); this is another way of saying that $(D-1)e^x = 0$. Applying $D-1$ to both sides of the DE, we get $(D-1)(y'' + xy) = y''' - y'' + xy' + (1-x)y = 0$, which is a homogeneous linear DE.
Just one word of caution: we can write the initial DE as $(D^2 + x)y = e^x$; however, we need to be careful when working with differential operators which involve both D and x. That's because $Dx \neq xD$, which we can see from $Dxy = xy' + y$ versus $xDy = xy'$. In other words, x and D don't commute (just like generic matrices). ◇

Example 124. Find the general solution of $\boldsymbol{x}' = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix} \boldsymbol{x}$.

Solution. The characteristic polynomial is

$$\begin{vmatrix} 1-\lambda & 1 & -1 \\ 1 & 1-\lambda & 1 \\ -1 & 1 & 1-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ -1 & 1-\lambda \end{vmatrix} - \begin{vmatrix} 1 & 1-\lambda \\ -1 & 1 \end{vmatrix} = (1-\lambda)^3 - 3(1-\lambda) - 2.$$

Since $x^3 - 3x - 2 = (x+1)^2(x-2)$, the eigenvalues are $\lambda = 1 - x = 2, 2, -1$. Note that $\lambda = 2$ is repeated! We say that the eigenvalue $\lambda = 2$ has multiplicity 2.

$$\boldsymbol{\lambda = -1.} \quad \left(\begin{array}{ccc|c} 2 & 1 & -1 & 0 \\ 1 & 2 & 1 & 0 \\ -1 & 1 & 2 & 0 \end{array}\right) \underset{\substack{2r_2 - r_1 \\ 2r_3 + r_1}}{\Longrightarrow} \left(\begin{array}{ccc|c} 2 & 1 & -1 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & 3 & 3 & 0 \end{array}\right) \Longrightarrow \left(\begin{array}{ccc|c} 2 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Setting, $v_3 = c$, we find $v_2 = -c$. Then, $2v_1 + v_2 - v_3$ implies $v_1 = c$. Setting $c = 1$, we find $\boldsymbol{v} = (1, -1, 1)^T$.

$$\boldsymbol{\lambda = 2.} \quad \left(\begin{array}{ccc|c} -1 & 1 & -1 & 0 \\ 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 0 \end{array}\right) \underset{\substack{r_2 + r_1 \\ r_3 - r_1}}{\Longrightarrow} \left(\begin{array}{ccc|c} -1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

The two zero rows are good news! It means that we will find two independent eigenvectors.
Indeed, we are free to set $v_3 = c$ and $v_2 = d$. Since $-v_1 + v_2 - v_3 = 0$, it follows that $v_1 = d - c$. Hence, the most general solution to the eigenvector equation is

$$\boldsymbol{v} = \begin{pmatrix} d - c \\ d \\ c \end{pmatrix} = c\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + d\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

Consequently, we have found solutions $\boldsymbol{x}_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} e^{-t}$, $\boldsymbol{x}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} e^{2t}$, $\boldsymbol{x}_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^{2t}$.

The Wronskian at 0 is $W(0) = \begin{vmatrix} 1 & -1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = -3 \neq 0$, which certifies that our three solutions are independent.

Hence, the general solution is $\boldsymbol{x} = c_1 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} e^{2t} + c_3 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^{2t}$. ◇