Theorem 106. Let $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n$ be solutions of $\boldsymbol{x}' = A(t)\,\boldsymbol{x}$. $A(t)$ is $n \times n$, entries continuous on I.

$$\begin{aligned} & c_1\boldsymbol{x}_1 + c_2\boldsymbol{x}_2 + \ldots + c_n\boldsymbol{x}_n \text{ is the general solution} \\ \iff\ & \boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n \text{ are independent} \\ \iff\ & \text{the Wronskian } W(t) = \det\left(\begin{array}{cccc} \boldsymbol{x}_1 & \boldsymbol{x}_2 & \ldots & \boldsymbol{x}_n \end{array}\right) \neq 0 \text{ for } t \in I \text{ of our choice} \end{aligned}$$

Moreover, such solutions always exist (on all of I).

Example 107. $x''' - 6x'' + 11x' - 6x = 0$ has solutions $x_1 = e^t$, $x_2 = e^{2t}$, $x_3 = e^{3t}$.

$$W(t) = \det\begin{pmatrix} e^t & e^{2t} & e^{3t} \\ e^t & 2e^{2t} & 3e^{3t} \\ e^t & 4e^{2t} & 9e^{3t} \end{pmatrix}, \quad W(0) = \det\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{pmatrix} = \det\begin{pmatrix} 2 & 3 \\ 4 & 9 \end{pmatrix} - \det\begin{pmatrix} 1 & 3 \\ 1 & 9 \end{pmatrix} + \det\begin{pmatrix} 1 & 2 \\ 1 & 4 \end{pmatrix} = 2 \neq 0.$$

This certifies[15] that our three solutions are indeed independent. As a consequence, the general solution is $c_1x_1 + c_2x_2 + c_3x_3$. ♢

Example 108. Introducing $y = x'$ and $z = x''$, the previous DE is equivalent to the first-order system $x' = y$, $y' = z$, $z' = 6x - 11y + 6z$. Writing $\boldsymbol{x} = (x, y, z)^T$, this system can be expressed as

$$\boldsymbol{x}' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -11 & 6 \end{pmatrix}\boldsymbol{x}.$$

The solutions from the previous example translate into $\boldsymbol{x}_1 = \begin{pmatrix} e^t \\ e^t \\ e^t \end{pmatrix}$, $\boldsymbol{x}_2 = \begin{pmatrix} e^{2t} \\ 2e^{2t} \\ 4e^{2t} \end{pmatrix}$, $\boldsymbol{x}_3 = \begin{pmatrix} e^{3t} \\ 3e^{3t} \\ 9e^{3t} \end{pmatrix}$.

The Wronskian is $W(t) = \det\left(\begin{array}{ccc} \boldsymbol{x}_1 & \boldsymbol{x}_2 & \boldsymbol{x}_3 \end{array}\right) = \det\begin{pmatrix} e^t & e^{2t} & e^{3t} \\ e^t & 2e^{2t} & 3e^{3t} \\ e^t & 4e^{2t} & 9e^{3t} \end{pmatrix}$. Exactly as before!

We again conclude that $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3$ are independent. The general solution is $c_1\boldsymbol{x}_1 + c_2\boldsymbol{x}_2 + c_3\boldsymbol{x}_3$. ♢

Example 109. Solve the initial value problem $\boldsymbol{x}' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -11 & 6 \end{pmatrix}\boldsymbol{x}$, $\boldsymbol{x}(0) = \begin{pmatrix} -2 \\ -1 \\ 3 \end{pmatrix}$.

Solution. From above, we know that the general solution is $\boldsymbol{x} = c_1\begin{pmatrix} e^t \\ e^t \\ e^t \end{pmatrix} + c_2\begin{pmatrix} e^{2t} \\ 2e^{2t} \\ 4e^{2t} \end{pmatrix} + c_3\begin{pmatrix} e^{3t} \\ 3e^{3t} \\ 9e^{3t} \end{pmatrix} = \begin{pmatrix} e^t & e^{2t} & e^{3t} \\ e^t & 2e^{2t} & 3e^{3t} \\ e^t & 4e^{2t} & 9e^{3t} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$. The matrix $\left(\begin{array}{ccc} \boldsymbol{x}_1 & \boldsymbol{x}_2 & \boldsymbol{x}_3 \end{array}\right)$ is called a fundamental matrix.

In order to solve the IVP, we have to solve $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \\ 3 \end{pmatrix}$.

$$\left.\begin{array}{ccc|c} 1 & 1 & 1 & -2 \\ 1 & 2 & 3 & -1 \\ 1 & 4 & 9 & 3 \end{array}\right. \implies \left.\begin{array}{ccc|c} 1 & 1 & 1 & -2 \\ 0 & 1 & 2 & 1 \\ 0 & 3 & 8 & 5 \end{array}\right. \implies \left.\begin{array}{ccc|c} 1 & 1 & 1 & -2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 2 & 2 \end{array}\right. \implies c_3 = 1, c_2 = -1, c_1 = -2$$

Hence, the solution to the IVP is $\boldsymbol{x}(t) = -2\begin{pmatrix} e^t \\ e^t \\ e^t \end{pmatrix} - \begin{pmatrix} e^{2t} \\ 2e^{2t} \\ 4e^{2t} \end{pmatrix} + \begin{pmatrix} e^{3t} \\ 3e^{3t} \\ 9e^{3t} \end{pmatrix} = \begin{pmatrix} -2e^t - e^{2t} + e^{3t} \\ -2e^t - 2e^{2t} + 3e^{3t} \\ -2e^t - 4e^{2t} + 9e^{3t} \end{pmatrix}$. ♢

We now turn to actually solving systems $\boldsymbol{x}' = A\boldsymbol{x}$ where A is a $n \times n$ matrix with constant entries.

Looking back at our examples so far, it makes sense to look for solutions of the form $\boldsymbol{x}(t) = \boldsymbol{v}e^{\lambda t}$ with $\boldsymbol{v}$ a vector which does not depend on t. Plugging into the DE, we get $\boldsymbol{x}' = \boldsymbol{v}\lambda e^{\lambda t} \overset{!}{=} Ae^{\lambda t}\boldsymbol{v}$. Cancelling the exponentials, we see that we have a solution if and only if $A\boldsymbol{v} = \lambda\boldsymbol{v}$.

For those familiar with the language of Linear Algebra this means that $\boldsymbol{v}$ is an eigenvector of A with eigenvalue λ.

15. Though we do know *a priori* that our method of solving HLDEs with constant coefficients will always produce independent solutions. It is good to see that the Wronskian agrees; it has to.