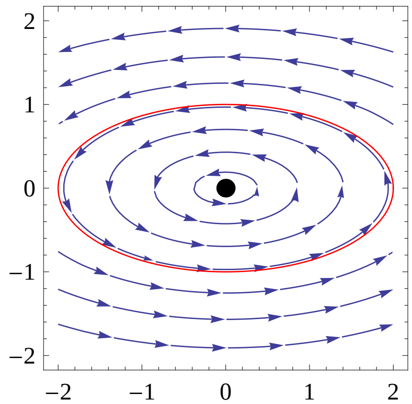
Review. systems of differential equations; express DEs as first-order systems ◇

Example 89. Express the non-linear DE $x'' = x^3 + (x')^3$ as a first-order system[14].

Solution. Introduce $x_1 = x$, $x_2 = x'$ to obtain the system $x_1' = x_2$, $x_2' = x_1^3 + x_2^3$. ◇

Example 90. Solve the system $x' = -2y$, $y' = \frac{1}{2}x$.



Solution. Observe that $x'' = -2y' = -x$. It follows that $x(t) = B_1\cos(t) + B_2\sin(t) = A\cos(t - \alpha)$ where $(B_1, B_2) = A(\cos(\alpha), \sin(\alpha))$. Consequently, $y = -\frac{1}{2}x' = \frac{1}{2}A\sin(t - \alpha)$.

Since $\cos^2 + \sin^2 = 1$, each solution $(x(t), y(t)) = \left(A\cos(t - \alpha), \frac{1}{2}A\sin(t - \alpha)\right)$ satisfies $\frac{x^2}{A^2} + \frac{y^2}{(A/2)^2} = 1$. This describes an ellipse! Several such curves are depicted in the phase plane portrait on the right which plots points (x, y) (also included are arrows to indicate evolution in time).

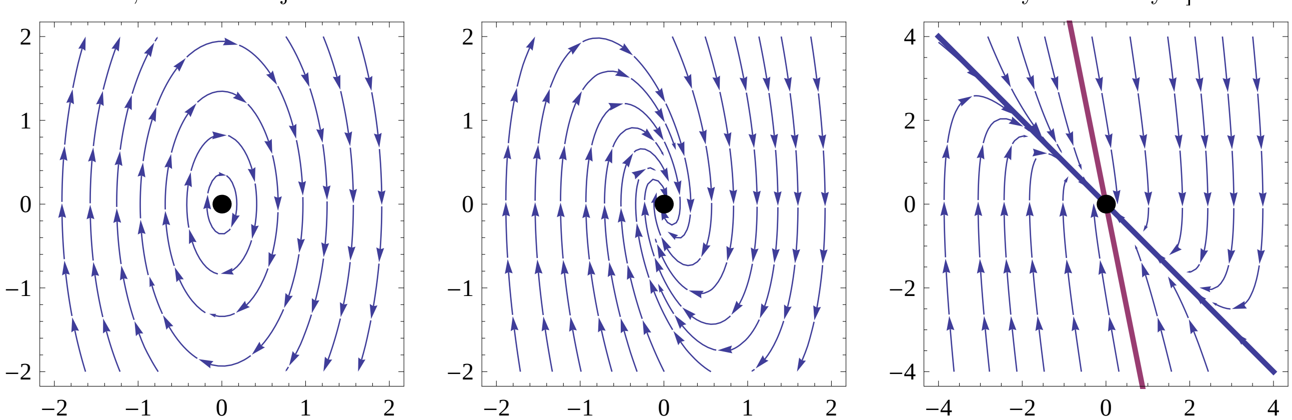
Note that we can use $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y/\mathrm{d}t}{\mathrm{d}x/\mathrm{d}t} = \frac{\frac{1}{2}x}{-2y} = -\frac{1}{4}\frac{x}{y}$ to sketch the phase portrait (just like for slope fields) without finding the solutions first. Do you also see how to get the directions? ◇

Example 91. Solve $x' = -2y$, $y' = \frac{1}{2}x$, $x(0) = 2$, $y(0) = 0$.

Solution. From before, $x(t) = B_1\cos(t) + B_2\sin(t)$ and $y(t) = -\frac{1}{2}x'(t) = \frac{1}{2}B_1\sin(t) - \frac{1}{2}B_2\cos(t)$. Hence, $x(0) = B_1 = 2$ and $y(0) = -B_2/2 = 0$. The solution $x(t) = 2\cos(t)$, $y(t) = \sin(t)$ is the red curve in the phase portrait. ◇

Example 92. Let $x'' + dx' + cx = 0$ describe the motion of a mass on a spring. Besides x (position), introduce $y = x'$ (velocity) to write it as the system $x' = y$, $y' = -cx - dy$.

- **undamped.** $x'' + 4x = 0$ (roots $\pm 2i$) translates into $x' = y$, $y' = -4x$. The solutions are $x = C\cos(2t - \alpha)$ and $y = x' = -2C\sin(2t - \alpha)$. As in the previous example, the points $(x(t), y(t))$ lie on an ellipse. Look at a trajectory of the phase portrait and describe the physical meaning of its path.
- **underdamped.** $x'' + 2x' + 5x = 0$ (roots $-1 \pm 2i$) translates into $x' = y$, $y' = -5x - 2y$. The solutions are $x = e^{-t}(B_1\cos(2t) + B_2\sin(2t))$ and $y = x' = e^{-t}((2B_2 - B_1)\cos(2t) - (2B_1 + B_2)\sin(2t))$. The points $(x(t), y(t))$ spiral towards the origin.
- **overdamped.** $x'' + 6x' + 5x = 0$ (roots $-1, -5$) translates into $x' = y$, $y' = -5x - 6y$. The solutions are $x = B_1e^{-t} + B_2e^{-5t}$ and $y = x' = -B_1e^{-t} - 5B_2e^{-5t}$. The points $(x(t), y(t))$ approach the origin. The trajectories are simple for the special solutions $(x, y) = (e^{-t}, -e^{-t}) = e^{-t}(1, -1)$ and $(x, y) = (e^{-5t}, -5e^{-5t}) = e^{-5t}(1, -5)$; in both cases, these are just lines. [Other solutions are a combination of these two, but the trajectories move much faster in direction of the second line. Do you see why?!] ◇



14. Such translations are useful for practical purposes. For instance, it means that one only needs to develop numerical algorithms for first-order systems.