External forces plus damping

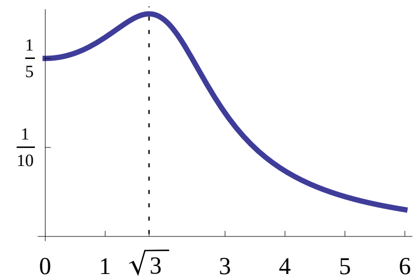
Example 85. Find the general solution of $2x'' + 2x' + x = 10\sin(t)$.

Solution. "Old" roots $\frac{-2 \pm \sqrt{4-8}}{4} = -\frac{1}{2} \pm \frac{1}{2}i$. So the system without external force is underdamped. [Why?!]
After a routine calculation, $x_p = -4\cos(t) - 2\sin(t) = \sqrt{20}(\cos(t-\alpha))$ with $\alpha = \tan^{-1}(1/2) + \pi \approx 3.605$. Here, we used that $(-4,-2) = \sqrt{20}(\cos\alpha, \sin\alpha)$.
Hence, the general solution is $x(t) = \underbrace{\sqrt{20}\cos(t-\alpha)}_{x_{\text{sp}}} + \underbrace{e^{-t/2}(c_1\cos(t/2) + c_2\sin(t/2))}_{x_{\text{tr}} \to 0 \text{ as } t \to \infty}$.

Observe how $x = x_{\text{tr}} + x_{\text{sp}}$ splits into transient motion x_{tr} and steady periodic oscillations x_{sp}. ◇

Example 86. Find the steady periodic solution to $x'' + 2x' + 5x = \cos(\omega t)$. What is the amplitude of the steady periodic oscillations? For which ω is the amplitude maximal?

Solution. "Old" roots $-1 \pm 2i$. [Not really needed, because positive damping prevents duplication; can you see it?]
Hence, $x_{\text{sp}} = A_1\cos(\omega t) + A_2\sin(\omega t)$ and to find A_1, A_2 we need to plug into the DE.
Doing so (we did!), we find $A_1 = \frac{5-\omega^2}{(5-\omega^2)^2 + 4\omega^2}$, $A_2 = \frac{2\omega}{(5-\omega^2)^2 + 4\omega^2}$.
Consequently, the amplitude of x_{sp} is $A_{\text{sp}} = \sqrt{A_1^2 + A_2^2} = \frac{1}{\sqrt{(5-\omega^2)^2 + 4\omega^2}}$.
The function $A_{\text{sp}}(\omega)$ is sketched to the right. It has a maximum at $\omega = \sqrt{3}$ at which the amplitude is unusually large (well, here it is not very pronounced). We say that practical resonance occurs for $\omega = \sqrt{3}$.
[For comparison, without damping, (pure) resonance occurs for $\omega = \sqrt{5}$.]



◇

Systems of differential equations

Example 87. Consider two springs attached to each other as in Figure 4.1.1.

Write $x_1(t)$ for the displacement of mass m_1 from equilibrium and, likewise, $x_2(t)$ for the mass m_2. Note that the first spring is stretched by x_1 whereas the second spring is stretched by $x_2 - x_1$. Applying Hooke's law and Newton's second law to each mass, while assuming the other one to be stationary, we find that

$$\begin{aligned} m_1 x_1'' &= -k_1 x_1 + k_2(x_2 - x_1), \\ m_2 x_2'' &= -k_2(x_2 - x_1). \end{aligned}$$

This is a system of differential equations. This particular one is linear and second-order.

Of course, now one can again introduce damping, external forces, etc. ◇

Fact. Any DE (or system) can be transformed into a first-order system of DEs! ◇

Example 88. Write $y''' + a(x)y'' + b(x)y' + c(x)y = f(x)$ as a first-order system.

Solution. Introduce $y_1 = y$, $y_2 = y'$, $y_3 = y''$. Then

$$\begin{aligned} y_1' &= y_2, \\ y_2' &= y_3, \\ y_3' &= -c(x)y_1 - b(x)y_2 - a(x)y_3 + f(x). \end{aligned}$$

This system is equivalent to the original DE in that y solves the original DE if and only if $(y_1, y_2, y_3) = (y, y', y'')$ solves the system of DEs.

By the way, the above system can be expressed in matrix-vector notation as

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -c(x) & -b(x) & -a(x) \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ f(x) \end{pmatrix}, \quad \text{or, for short,} \quad \boldsymbol{y}' = A(x)\boldsymbol{y} + F(x),$$

where, in the final expression, $A(x)$ is the 3×3 matrix and $F(x)$ the vector. This is not just cosmetics but understanding matrices will allow us to use similar techniques as before; in many ways, we will be able to treat A just like a number. ◇

Armin Straub
astraub@illinois.edu