Theorem 74. The linear DE $Ly = y'' + P(x)y' + Q(x)y = f(x)$ has particular solution

$$y_p = -y_1(x)\int \frac{y_2(x)f(x)}{W(x)}\mathrm{d}x + y_2(x)\int \frac{y_1(x)f(x)}{W(x)}\mathrm{d}x,$$

where y_1, y_2 are independent solutions of $Ly = 0$ and $W = y_1y_2' - y_1'y_2$ is the Wronskian of y_1, y_2.
[Note that considering all possible constants of integration actually gives the general solution of $Ly = f(x)$.]

Proof. Let us look for $y_p = u_1(x)\,y_1(x) + u_2(x)\,y_2(x)$. This "ansatz" is called variation of constants/parameters.
Then $y_p' = \underbrace{u_1'y_1 + u_2'y_2}_{=0 \text{ (or so we wish)}} + u_1y_1' + u_2y_2'$ and $y_p'' = u_1'y_1' + u_2'y_2' + u_1y_1'' + u_2y_2''$.

$Ly_p = u_1'y_1' + u_2'y_2' + u_1y_1'' + u_2y_2'' + P(x)(u_1y_1' + u_2y_2') + Q(x)(u_1y_1 + u_2y_2)$
$= u_1'y_1' + u_2'y_2' + u_1Ly_1 + u_2Ly_2 = u_1'y_1' + u_2'y_2'$

So, in order for $y_p = u_1y_1 + u_2y_2$ to solve $Ly = f(x)$, we need

$$\begin{aligned} u_1'y_1 + u_2'y_2 &= 0, \\ u_1'y_1' + u_2'y_2' &= f(x). \end{aligned}$$

These are linear equations in u_1' and u_2'. Solving gives $u_1' = \frac{-y_2\,f(x)}{y_1y_2' - y_1'\,y_2}$ and $u_2' = \frac{y_1\,f(x)}{y_1y_2' - y_1'\,y_2}$, and it only remains to integrate. □

Example 75. Find a particular solution of $y'' - 2y' + y = \frac{e^x}{x}$.

Solution. Here, $y_1 = e^x$, $y_2 = xe^x$. We calculate $W(x) = e^{2x}$.
$y_p = -e^x\int 1\mathrm{d}x + xe^x\int \frac{1}{x}\mathrm{d}x = xe^x[\ln|x| - 1]$. (Note that, with integration constants, we get $-e^x(x + C_1) + xe^x(\ln|x| + C_2)$, which is the general solution. So any constants suffice to give us a particular solution.) ◇

Example 76. Solve $Ly = x^2y'' - 4xy' + 6y = x^3$. Given: $y_1 = x^2$ and $y_2 = x^3$ solve $Ly = 0$.

Solution. First, $W(x) = x^4$. Put DE in the form $y'' - 4x^{-1}y' + 6x^{-2}y = x$.

$$y_p = -y_1(x)\int \frac{y_2(x)f(x)}{W(x)}\mathrm{d}x + y_2(x)\int \frac{y_1(x)f(x)}{W(x)}\mathrm{d}x = -x^2\int 1\,\mathrm{d}x + x^3\int \frac{1}{x}\mathrm{d}x = -x^3 + x^3\ln|x|.$$

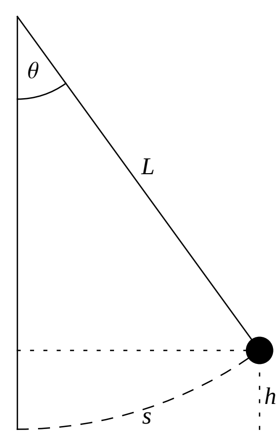
Hence, the general solution is $C_1x^2 + (C_2 + \ln|x|)x^3$. ◇

Remark 77. Just for fun (and understanding and context), let us revisit our method of solving first-order linear DEs $Ly = y' + P(x)y = f(x)$. [Variation of constants also extends to higher-order DEs.]

Note that $Ly = 0$ has solution $y_1 = \exp(-\int P(x)\mathrm{d}x)$, which is nothing but the inverse of the integrating factor!
Using the integrating factor, we arrive at $y_p(x) = y_1(x)\int \frac{f(x)}{y_1(x)}\mathrm{d}x$ which is the analog of Theorem 74. ◇

Application: motion of a pendulum

The motion of an (ideal) pendulum is described by $\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \frac{g}{L}\sin\theta = 0$, where θ is the angular displacement and L is the length of the pendulum (and, as usual, g is acceleration due to gravity).



Proof. We assume the string to be massless, and let m be the swinging mass.
Let s and h be as in the picture on the right.
Velocity (more accurately, speed) of mass: $v = \frac{\mathrm{d}s}{\mathrm{d}t} = L\frac{\mathrm{d}\theta}{\mathrm{d}t}$
Kinetic energy: $T = \frac{1}{2}mv^2 = \frac{1}{2}mL^2\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2$
Potential energy: $V = mgh = mgL(1 - \cos\theta)$ (weight mg times height h)
Conservation of energy: $T + V = \text{const}$
Take time derivative: $\frac{1}{2}mL^2\,2\frac{\mathrm{d}\theta}{\mathrm{d}t}\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + mgL\sin\theta\frac{\mathrm{d}\theta}{\mathrm{d}t} = 0$. Finally, cancel terms. □

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