Review. homogeneous linear DEs with constant coefficients ◇

Example 43. Find the general solution of $y''' - y'' - 5y' - 3y = 0$.

Solution. The characteristic equation is $r^3 - r^2 - 5r - 3 = (r-3)(r+1)^2$.
This corresponds to the solutions $y_1 = e^{3x}$, $y_2 = e^{-x}$, $y_3 = xe^{-x}$.
Hence, the general solution is $y(x) = Ae^{3x} + (B + Cx)\, e^{-x}$. ◇

Complex roots and complex exponentials

Example 44. Find the general solution of $y'' + y = 0$.

Solution. The characteristic equation is $r^2 + 1 = 0$ which has no solutions over the reals.
Over the complex numbers, by definition, the roots are i and $-i$.
So the general solutions is $y(x) = Ae^{ix} + Be^{-ix}$.

Solution. On the other hand, we easily check that $y_1 = \cos(x)$ and $y_2 = \sin(x)$ are two solutions.
Hence, the general solution can also be written as $y(x) = C\cos(x) + D\sin(x)$. ◇

Example 45. What is going on in the previous example?

To compare specific functions, let us consider initial values. Then, e^{ix} is the unique solution of $y'' + y = 0$ which satisfies $y(0) = 1$ and $y'(0) = i$. On the other hand, solving the IVP using $y(x) = C\cos(x) + D\sin(x)$, we get $C = 1$ and $D = i$. This shows the fundamental identity

$$e^{ix} = \cos(x) + i\sin(x),$$

known as Euler's identity. ◇

Remark 46. Setting $x = \pi$ in Euler's identity and rearranging, we get $e^{i\pi} + 1 = 0$, which combines the five most important mathematical constants in a single beautiful formula. ◇

Definition 47. Any complex number $z \in \mathbb{C}$ can be written as $z = x + iy$, with $x, y \in \mathbb{R}$. x is called the real part and y the imaginary part. The complex conjugate of z is $\bar{z} = x - iy$.

From abc-formula: if $z = x + iy$ is the root of a polynomial (with real coefficients), then so is $\bar{z} = x - iy$.

Its absolute value is $r = |z| = \sqrt{x^2 + y^2}$. Its argument (or amplitude or phase) is the angle θ from the positive real axis to the vector (x, y) representing z.

In fact, this gives the polar form $z = x + iy = re^{i\theta}$. [by Euler's identity!]

Example 48. Find the general solution of $y'' + 4y' + 13y = 0$.

Solution. The characteristic polynomial is $r^2 + 4r + 13 = (r - (-2+3i))(r - (-2+3i))$.
$y_1 = e^{(-2+3i)x} = e^{-2x}e^{3ix} = e^{-2x}(\cos(3x) + i\sin(3x))$, $y_2 = e^{(-2-3i)x} = e^{-2x}e^{-3ix} = e^{-2x}(\cos(3x) - i\sin(3x))$
Note that $\frac{1}{2}(y_1 + y_2) = e^{-2x}\cos(3x)$ and $\frac{1}{2i}(y_1 - y_2) = e^{-2x}\sin(3x)$ are solutions as well. And they are real!
So, the general solution is $Ae^{-2x}\cos(3x) + Be^{-2x}\sin(3x)$. This always works! ◇

Theorem 49. Consider, again, a homogeneous linear DE with constant coefficients.

- If r_0 is a root of the characteristic polynomial and if k is its multiplicity, then $e^{r_0 x}$, $xe^{r_0 x}$, ..., $x^{k-1}e^{r_0 x}$ are solutions of the DE.
- If $r_0 = a + bi$ is a complex root, then $a - bi$ is another root, and we can write the corresponding solutions as $e^{ax}\cos(bx)$ and $e^{ax}\sin(bx)$.

  If the roots are repeated, we again have $x^j e^{ax}\cos(bx)$ and $x^j e^{ax}\sin(bx)$ as additional solutions.

Example 50. Find the general solution of $y^{(7)} + 8y^{(6)} + 42y^{(5)} + 104y^{(4)} + 169y''' = 0$.

Solution. The characteristic polynomial factors as $r^3(r^2 + 4r + 13)^2 = r^3(r - (-2+3i))^2(r - (-2+3i))^2$.
Hence, the general solution is $(A + Bx + Cx^2) + (D + Ex)\, e^{-2x}\cos(3x) + (F + Gx)e^{-2x}\sin(3x)$. ◇

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