Example 38. Solve the IVP $y''' + 7y'' + 14y' + 8y = 0$ with $y(0) = 1$, $y'(0) = 0$, $y''(0) = 1$.

Solution. Last time, we found that the DE has the general solution $y(x) = Ae^{-x} + Be^{-2x} + Ce^{-4x}$.
$y(x) = Ae^{-x} + Be^{-2x} + Ce^{-4x}$, $y(0) = A + B + C = 1$
$y'(x) = -Ae^{-x} - 2Be^{-2x} - 4Ce^{-4x}$, $y'(0) = -A - 2B - 4C = 0$
$y''(x) = Ae^{-x} + 4Be^{-2x} + 16Ce^{-4x}$, $y''(0) = A + 4B + 16C = 1$
Solving the system of linear equations, we find $A = 3, B = -5/2, C = 1/2$. Hence, the solution to the IVP is $y(x) = 3e^{-x} - 5/2e^{-2x} + 1/2e^{-4x}$. ◇

Example 39. Consider the IVP from the previous example.

Note that the DE let's us determine $y'''(0) = -7y''(0) - 14y'(0) - 8y(0) = -15$ (without solving it!). By applying $\frac{\mathrm{d}}{\mathrm{d}x}$ to the DE, we can likewise find $y^{(4)}(0)$, $y^{(5)}(0)$, ...
This can be done with any DE and gives another indication why an IVP "usually" has a unique solution, and why initial conditions of this form are very natural to consider. ◇

Example 40. Find the general solution of $y'' = 0$. [Then, $y^{(n)} = 0$.]

Solution. We know from Calculus that the general solution is $y(x) = A + Bx$.

Solution. The characteristic equation is $r^2 = 0$. So one solution is $y_1 = e^{0x} = 1$. But what is a second solution? As Calculus showed, a second solution is $y_2 = xe^{0x} = x$. It turns out that this always works! ◇

Example 41. Find the general solution of $y'' - 2y' + y = 0$.

Solution. The characteristic equation is $r^2 - 2r + 1 = (r-1)^2$. Hence, $y_1 = e^x$.
But what is the second solution? Inspired by the previous example, we can check that $y_2 = xe^x$ is a solution. Hence, the general solution is $y(x) = Ae^x + Bxe^x$. ◇

Theorem 42. Consider a homogeneous linear DE with constant coefficients $y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1y' + a_0y = 0$. (Its characteristic polynomial is $p(r) = r^n + a_{n-1}r^{n-1} + \ldots + a_1r + a_0$.)

- If r_0 is a root of the characteristic polynomial and if k is its multiplicity (this means that $(r - r_0)^k$ is a factor of $p(r)$), then e^{r_0x}, xe^{r_0x}, ..., $x^{k-1}e^{r_0x}$ are solutions of the DE.
- Combining these solutions for all roots r_0, actually gives the general solution.

  This is because a polynomial of degree n has (counting with multiplicity) exactly n (possibly complex) roots. More on complex number in due time.

Proof. Set $D = \frac{\mathrm{d}}{\mathrm{d}x}$. A homogeneous linear DE with constant coefficients can be written as $p(D)y = 0$, where $p(D)$ is a polynomial in D. [For instance, $y'' - 2y' + y = 0$ is $D^2y - 2Dy + y = (D^2 - 2D + 1)y = (D-1)^2y = 0$.]
In fact, we see that $p(r)$ is just the characteristic polynomial!
If r_0 is a root of the characteristic polynomial, then $p(r) = q(r)\,(r - r_0)^k$, where $k \geqslant 1$ is its multiplicity.
The DE factors likewise and can be written as $q(D)\,(D - r_0)^k y = 0$.
From here we see that solutions to $(D - r_0)^k y = 0$ will solve our original DE.
Let $y(x)$ be a solution of $(D - r_0)^k y = 0$. Write it as $y(x) = u(x)e^{r_0x}$ (we can always do that for some $u(x)$).
Let $u(x)$ be some function. Note that $(D - r_0)[ue^{r_0x}] = u'e^{r_0x} + ur_0e^{r_0x} - r_0[ue^{r_0x}] = u'e^{r_0x}$.
Repeating, we get $(D - r_0)^2[ue^{r_0x}] = (D - r_0)[u'e^{r_0x}] = u''e^{r_0x}$ and, eventually, $(D - r_0)^k[ue^{r_0x}] = u^{(k)}e^{r_0x}$. In particular, $(D - r_0)^k y = 0$ is solved by $y = ue^{r_0x}$ if $u^{(k)} = 0$.
This latter condition gives $u(x) = C_0 + C_1x + \ldots + C_{k-1}x^{k-1}$ and it follows that $y(x) = (C_0 + C_1x + \ldots + C_{k-1}x^{k-1})e^{r_0x}$ solves our original DE, as claimed. □