Review. linear first-order equations ◇

Example 25. A tank contains 20gal of pure water. It is filled with brine (containing 2lb/gal salt) at a rate of 3gal/min. At the same time, well-mixed solution flows out at a rate of 2gal/min. How much salt is in the tank after t minutes?

Solution. volume (in gal) in tank after time t (in min): $V(t) = 20 + (3-2)t = 20 + t$

amount of salt (in lb) in tank: $x(t)$

concentration of salt (in lb/gal) in tank: $\frac{x(t)}{V(t)}$

In the time interval $[t, t + \Delta t]$, $\Delta x \approx 3 \cdot 2 \cdot \Delta t - 2 \cdot \frac{x(t)}{V(t)} \cdot \Delta t$.

Hence, x solves the IVP $\frac{\mathrm{d}x}{\mathrm{d}t} = 6 - 2 \cdot \frac{x}{20+t}$, $x(0) = 0$. This is a linear DE!

$\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{2}{20+t} x = 6$. The integrating factor is $f(t) = e^{\int \frac{2}{20+t}\mathrm{d}t} = (20+t)^2$.

$(20+t)^2 \frac{\mathrm{d}x}{\mathrm{d}t} + 2(20+t)x = \frac{\mathrm{d}}{\mathrm{d}t}[(20+t)^2 x] = \frac{\mathrm{d}}{\mathrm{d}t}[\int 6(20+t)^2 \mathrm{d}t] = \frac{\mathrm{d}}{\mathrm{d}t}[2(20+t)^3]$

Hence, $(20+t)^2 x = 2(20+t)^3 + C$. Using $x(0) = 0$, we find $C = -2 \cdot 20^3$.

This means that, after t minutes, the tank contains $x(t) = 2(20+t) - \frac{2 \cdot 20^3}{(20+t)^2}$ pounds of salt.

As a consequence, we get that $x(t) \approx 2(20+t) = 2V(t)$ for large t. Why does that make perfect sense?! ◇

Substitutions

Example 26. Solve $\frac{\mathrm{d}y}{\mathrm{d}x} = (x+y)^2$.

Solution. Set $u = x + y$. Then $\frac{\mathrm{d}u}{\mathrm{d}x} = 1 + \frac{\mathrm{d}y}{\mathrm{d}x}$ and, hence, $\frac{\mathrm{d}u}{\mathrm{d}x} - 1 = u^2$.

This DE for u can be solved by separation of variables: $\frac{1}{1+u^2}\mathrm{d}u = \mathrm{d}x$, $\arctan(u) = x + C$, $u = \tan(x + C)$.

The solution of the original DE is $y = u - x = \tan(x + C) - x$. ◇

Useful substitutions

Some important, easy-to-spot, cases:

- $y' = F\left(\frac{y}{x}\right)$. This is called a homogeneous equation.

  Set $u = \frac{y}{x}$. Then $y = ux$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = x\frac{\mathrm{d}u}{\mathrm{d}x} + u$. We get $x\frac{\mathrm{d}u}{\mathrm{d}x} + u = F(u)$. This is always separable.
- $F(y'', y', x) = 0$. 2nd order with "y missing".

  Set $u = y' = \frac{\mathrm{d}y}{\mathrm{d}x}$. Then $y'' = \frac{\mathrm{d}u}{\mathrm{d}x}$. We get the first-order DE $F\left(\frac{\mathrm{d}u}{\mathrm{d}x}, u, x\right) = 0$.
- $F(y'', y', y) = 0$. 2nd order with "x missing".

  Set $u = y' = \frac{\mathrm{d}y}{\mathrm{d}x}$. Then $y'' = \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}y} \cdot \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}y} \cdot u$. We get the first-order DE $F\left(u\frac{\mathrm{d}u}{\mathrm{d}y}, u, y\right) = 0$.

Examples follow next time...

Armin Straub
astraub@illinois.edu