

Midterm #2 – Practice

MATH 238 — Differential Equations I

Midterm: Wednesday, Nov 12, 2025

Please print your name:

Bonus challenge. Let me know about any typos you spot in the posted solutions (or lecture sketches). Any mathematical typo, that is not yet fixed by the time you send it to me, is worth a bonus point.

Reminder. No notes, calculators or tools of any kind will be permitted on the midterm exam.

Problem 1. Let L be a linear differential operator of order 4 with constant real coefficients. Suppose that $3 + 7i$ is a repeated characteristic root of L .

- (a) What is the general solution to $Ly = 0$?
- (b) Write down the simplest form of a particular solution y_p of the DE $Ly = 7x^2e^{3x}$ with undetermined coefficients.
- (c) Write down the simplest form of a particular solution y_p of the DE $Ly = e^{3x}\sin(7x) + 3x^2$ with undetermined coefficients.

Problem 2.

- (a) Consider a homogeneous linear differential equation with constant real coefficients which has order 8. Suppose $y(x) = 7x - 2x^2e^{3x}\sin(5x)$ is a solution. Write down the general solution.
- (b) Consider a homogeneous linear differential equation with constant real coefficients which has order 8. Suppose $y(x) = 2xe^{3x} + x\cos(5x) - 5\sin(x)$ is a solution. Write down the general solution.
- (c) Write down a homogeneous linear differential equation satisfied by $y(x) = 1 - 5x^2e^{-2x}$.
Here, and elsewhere, you can use the operator D to write the DE. No need to simplify, any form is acceptable.
- (d) Write down a homogeneous linear differential equation satisfied by $y(x) = 2 - 3x(e^{4x} + e^{-4x}) - (7x^2 + 5)e^x$.
- (e) Let y_p be any solution to the inhomogeneous linear differential equation $y'' - 9y = 4xe^x - 5e^{2x}$. Find a homogeneous linear differential equation which y_p solves.

Problem 3.

- (a) Determine the general solution of the system $\begin{matrix} y_1' = y_1 - 6y_2 \\ y_2' = y_1 - 4y_2 \end{matrix}$.
- (b) Solve the IVP $\begin{matrix} y_1' = y_1 - 6y_2 \\ y_2' = y_1 - 4y_2 \end{matrix}$ with $\begin{matrix} y_1(0) = 4 \\ y_2(0) = 1 \end{matrix}$.
- (c) Determine a particular solution to $\begin{matrix} y_1' = y_1 - 6y_2 \\ y_2' = y_1 - 4y_2 - 2e^{3x} \end{matrix}$.
- (d) Determine the general solution to $\begin{matrix} y_1' = y_1 - 6y_2 \\ y_2' = y_1 - 4y_2 - 2e^{3x} \end{matrix}$.

Problem 4.

- (a) Write the (third-order) differential equation $y''' + 2y'' - 4y' + 5y = 2\sin(x)$ as a system of (first-order) differential equations.
- (b) Consider the following system of (second-order) initial value problems:

$$\begin{aligned} y_1'' &= 5y_1' + 2y_2' + e^{2x} \\ y_2'' &= 7y_1 - 3y_2 - 3e^x \end{aligned} \quad y_1(0) = 1, \quad y_1'(0) = 4, \quad y_2(0) = 0, \quad y_2'(0) = -1$$

Write it as a first-order initial value problem in the form $\mathbf{y}' = M\mathbf{y} + \mathbf{f}$, $\mathbf{y}(0) = \mathbf{c}$.

Problem 5. The mixtures in three tanks T_1, T_2, T_3 are kept uniform by stirring. Brine containing 2 lb of salt per gallon enters the first tank at a rate of 15 gal/min. Mixed solution from T_1 is pumped into T_2 at a rate of 10 gal/min and from T_2 into T_3 at a rate of 10 gal/min. Each tank initially contains 10 gal of pure water. Denote by $y_i(t)$ the amount (in pounds) of salt in tank T_i at time t (in minutes). Derive a system of linear differential equations for the y_i , including initial conditions.

Problem 6.

- (a) What is the period and the amplitude of $3\cos(7t) - 5\sin(7t)$?
- (b) Assume that the angle $\theta(t)$ of a swinging pendulum is described by $\theta'' + 4\theta = 0$. Suppose $\theta(0) = \frac{3}{10}$ and $\theta'(0) = -\frac{4}{5}$. What is the period and the amplitude of the resulting oscillations?
- (c) The position $y(t)$ of a certain mass on a spring is described by $y'' + dy' + 5y = 0$. For which value of $d > 0$ is the system underdamped? Critically damped? Overdamped?
- (d) A forced mechanical oscillator is described by $y'' + 2y' + y = 25\cos(2t)$. As $t \rightarrow \infty$, what is the period and the amplitude of the resulting oscillations?
- (e) The motion of a certain mass on a spring is described by $y'' + y' + \frac{1}{2}y = 5\sin(t)$ with $y(0) = 2$ and $y'(0) = 0$. Determine $y(t)$. As $t \rightarrow \infty$, what are the period and amplitude of the oscillations?

Problem 7. The position $y(t)$ of a certain mass on a spring is described by $2y'' + dy' + 3y = F(t)$.

- (a) Assume first that there is no external force, i.e. $F(t) = 0$. For which values of d is the system overdamped?
- (b) Now, $F(t) = \sin(4\omega t)$ and the system is undamped, i.e. $d = 0$. For which values of ω , if any, does resonance occur?
- (c) Now, $F(t) = 5\cos(\omega t) - 2\sin(3\omega t)$ and the system is undamped, i.e. $d = 0$. For which values of ω , if any, does resonance occur?

Problem 8.

- (a) Determine the general solution to $y'' - 4y' + 4y = 3e^{2x}$.

- (b) Determine the general solution to the differential equation $y''' - y = e^x + 7$.
- (c) Determine the general solution $y(x)$ to the differential equation $y^{(4)} + 6y''' + 13y'' = 2$. Express the solution using real numbers only.
- (d) Solve the initial value problem $y'' + 2y' + y = 2e^{2x} + e^{-x}$, $y(0) = -1$, $y'(0) = 2$.

Problem 9.

- (a) Consider the differential equation $x^2y'' - 4xy' + 6y = 0$. Find all solutions of the form $y(x) = x^r$.
- (b) Determine the general solution of $x^2y'' - 4xy' + 6y = x^3$.