

Midterm #2 – Practice

Please print your name:

Bonus challenge. Let me know about any typos you spot in the posted solutions (or lecture sketches). Any mathematical typo, that is not yet fixed by the time you send it to me, is worth a bonus point.

Reminder. No notes, calculators or tools of any kind will be permitted on the midterm exam.

Problem 1. Let L be a linear differential operator of order 4 with constant real coefficients. Suppose that $3 + 7i$ is a repeated characteristic root of L .

- (a) What is the general solution to $Ly = 0$?
- (b) Write down the simplest form of a particular solution y_p of the DE $Ly = 7x^2e^{3x}$ with undetermined coefficients.
- (c) Write down the simplest form of a particular solution y_p of the DE $Ly = e^{3x}\sin(7x) + 3x^2$ with undetermined coefficients.

Solution.

- (a) Since L is real, if $3 + 7i$ is a repeated characteristic root of L , then $3 - 7i$ must be a repeated characteristic root of L as well. Hence, the 4 characteristic roots must be $3 \pm 7i, 3 \pm 7i$.

The corresponding general solution is $(C_1 + C_2x)e^{3x}\cos(7x) + (C_3 + C_4x)e^{3x}\sin(7x)$.

- (b) The characteristic roots of the homogeneous DE are $3 \pm 7i, 3 \pm 7i$ while the roots for the inhomogeneous part are $3, 3, 3$.

Hence, there must a particular solution of the form $y_p = (C_1 + C_2x + C_3x^2)e^{3x}$.

The unique values of C_1, C_2, C_3 for which this is a solution of the DE need to be determined by plugging into the DE.

- (c) The characteristic roots of the homogeneous DE are $3 \pm 7i, 3 \pm 7i$ while the roots for the inhomogeneous part are $3 \pm 7i, 0, 0, 0$.

Hence, there must a particular solution of the form $C_1x^2e^{3x}\cos(7x) + C_2x^2e^{3x}\sin(7x) + C_3 + C_4x + C_5x^2$.

The unique values of C_1, C_2, C_3, C_4, C_5 for which this is a solution of the DE need to be determined by plugging into the DE.

Problem 2.

- (a) Consider a homogeneous linear differential equation with constant real coefficients which has order 8. Suppose $y(x) = 7x - 2x^2e^{3x}\sin(5x)$ is a solution. Write down the general solution.
- (b) Consider a homogeneous linear differential equation with constant real coefficients which has order 8. Suppose $y(x) = 2xe^{3x} + x\cos(5x) - 5\sin(x)$ is a solution. Write down the general solution.
- (c) Write down a homogeneous linear differential equation satisfied by $y(x) = 1 - 5x^2e^{-2x}$.
Here, and elsewhere, you can use the operator D to write the DE. No need to simplify, any form is acceptable.
- (d) Write down a homogeneous linear differential equation satisfied by $y(x) = 2 - 3x(e^{4x} + e^{-4x}) - (7x^2 + 5)e^x$.
- (e) Let y_p be any solution to the inhomogeneous linear differential equation $y'' - 9y = 4xe^x - 5e^{2x}$. Find a homogeneous linear differential equation which y_p solves.

Solution.

- (a) The characteristic roots must include $0, 0, 3 \pm 5i, 3 \pm 5i, 3 \pm 5i$. Since these are 8 roots and the DE has order 8, there cannot be any additional roots.

Hence, the general solution is $C_1 + C_2x + (C_3 + C_4x + C_5x^2)e^{3x}\cos(5x) + (C_6 + C_7x + C_8x^2)e^{3x}\sin(5x)$.

- (b) The characteristic roots must include $3, 3, \pm 5i, \pm 5i, \pm i$. Since these are 8 roots and the DE has order 8, there cannot be any additional roots.

Hence, the general solution is $(C_1 + C_2x)e^{3x} + (C_3 + C_4x)\cos(5x) + (C_5 + C_6x)\sin(5x) + C_7\cos(x) + C_8\sin(x)$.

- (c) $y(x) = 1 - 5x^2e^{-2x}$ is a solution of $p(D)y = 0$ if and only if $-2, -2, -2, 0$ are roots of the characteristic polynomial $p(D)$. Hence, the simplest DE is obtained from $p(D) = D(D+2)^3$.

Comment. If we wanted to, we could multiply out $D(D+2)^3 = D^4 + 6D^3 + 12D^2 + 8D$ and write the DE as $y^{(4)} + 6y''' + 12y'' + 8y' = 0$. However, this is usually neither needed nor useful.

- (d) In order for $y(x)$ to be a solution of $p(D)y = 0$, the characteristic roots must include $0, 4, 4, -4, -4, 1, 1, 1$.

Hence, the simplest differential equation is $D(D-4)^2(D+4)^2(D-1)^3y = 0$.

- (e) $(D-1)^2(D-2)(D^2-9)y = 0$

Explanation. Since y_p solves the inhomogeneous DE, we have $(D^2-9)y_p = 4xe^x - 5e^{2x}$. The right-hand side $4xe^x - 5e^{2x}$ is a solution of $p(D)y = 0$ if and only if $1, 1, 2$ are roots of the characteristic polynomial $p(D)$. In particular, $(D-1)^2(D-2)(4xe^x - 5e^{2x}) = 0$. Combined, we find that $(D-1)^2(D-2)(D^2-9)y_p = 0$.

Problem 3.

- (a) Determine the general solution of the system $\begin{aligned} y_1' &= y_1 - 6y_2 \\ y_2' &= y_1 - 4y_2 \end{aligned}$.

- (b) Solve the IVP $\begin{aligned} y_1' &= y_1 - 6y_2 \\ y_2' &= y_1 - 4y_2 \end{aligned}$ with $\begin{aligned} y_1(0) &= 4 \\ y_2(0) &= 1 \end{aligned}$.

- (c) Determine a particular solution to $\begin{aligned} y_1' &= y_1 - 6y_2 \\ y_2' &= y_1 - 4y_2 - 2e^{3x} \end{aligned}$.

- (d) Determine the general solution to $\begin{aligned} y_1' &= y_1 - 6y_2 \\ y_2' &= y_1 - 4y_2 - 2e^{3x} \end{aligned}$.

Solution.

- (a) Using $y_2 = \frac{1}{6}(y_1 - y_1')$ (from the first equation) in the second equation, we get $\frac{1}{6}(y_1' - y_1'') = y_1 - \frac{4}{6}(y_1 - y_1')$.

Simplified (and both sides multiplied by -6), this is $y_1'' + 3y_1' + 2y_1 = 0$.

This is a homogeneous linear DE with constant coefficients. The characteristic roots are $-1, -2$.

Hence, $y_1 = C_1e^{-x} + C_2e^{-2x}$.

We can then determine y_2 as $y_2 = \frac{1}{6}(y_1 - y_1') = \frac{1}{6}(C_1e^{-x} + C_2e^{-2x} - (-C_1e^{-x} - 2C_2e^{-2x})) = \frac{1}{3}C_1e^{-x} + \frac{1}{2}C_2e^{-2x}$.

- (b) From the previous part, we know $y_1 = C_1e^{-x} + C_2e^{-2x}$ and $y_2 = \frac{1}{3}C_1e^{-x} + \frac{1}{2}C_2e^{-2x}$.

We solve for C_1 and C_2 using the initial conditions: $y_1(0) = C_1 + C_2 \stackrel{!}{=} 4$ and $y_2(0) = \frac{1}{3}C_1 + \frac{1}{2}C_2 \stackrel{!}{=} 1$.

Solving these two equations, we find $C_1 = 6$ and $C_2 = -2$.

Thus, the unique solution to the IVP is $y_1 = 6e^{-x} - 2e^{-2x}$ and $y_2 = 2e^{-x} - e^{-2x}$.

- (c) We proceed as in the first part to write $y_2 = \frac{1}{6}(y_1 - y_1')$.

Using this in the second equation and simplifying, we get $y_1'' + 3y_1' + 2y_1 = 12e^{3x}$.

This is an inhomogeneous linear DE with constant coefficients. Since the characteristic roots of the homogeneous DE are $-1, -2$, while the root for the inhomogeneous part is 3 , there must a particular solution of the form $y_1 = Ce^{3x}$ with undetermined coefficient C . To determine C , we plug this y_1 into the DE: $y_1'' + 3y_1' + 2y_1 = (9 + 3 \cdot 3 + 2)Ce^{3x} = 20Ce^{3x} \stackrel{!}{=} 12e^{3x}$. Hence, $C = \frac{3}{5}$.

Having found $y_1 = \frac{3}{5}e^{3x}$, we can then determine y_2 as $y_2 = \frac{1}{6}(y_1 - y_1') = \frac{1}{6}(\frac{3}{5}e^{3x} - \frac{9}{5}e^{3x}) = -\frac{1}{5}e^{3x}$.

- (d) We get the general solution by adding the particular solution (previous part) and the general solution to the corresponding homogeneous equation (first part):

Hence, the general solution is $y_1 = \frac{3}{5}e^{3x} + C_1e^{-x} + C_2e^{-2x}$ and $y_2 = -\frac{1}{5}e^{3x} + \frac{1}{3}C_1e^{-x} + \frac{1}{2}C_2e^{-2x}$.

Alternatively. Here is a solution that proceeds from scratch (rather than referring to previous parts):

Using $y_2 = \frac{1}{6}(y_1 - y_1')$ (from the first equation) in the second equation, we get $\frac{1}{6}(y_1' - y_1'') = y_1 - \frac{4}{6}(y_1 - y_1') - 2e^{3x}$.

Simplified (and both sides multiplied by -6), this is $y_1'' + 3y_1' + 2y_1 = 12e^{3x}$.

This is an inhomogeneous linear DE with constant coefficients. Since the characteristic roots of the homogeneous DE are $-1, -2$, while the root for the inhomogeneous part is 3 , there must a particular solution of the form $y_1 = Ce^{3x}$ with undetermined coefficient C . To determine C , we plug this y_1 into the DE: $y_1'' + 3y_1' + 2y_1 = (9 + 3 \cdot 3 + 2)Ce^{3x} = 20Ce^{3x} \stackrel{!}{=} 12e^{3x}$. Hence, $C = \frac{3}{5}$ and the particular solution is $y_1 = \frac{3}{5}e^{3x}$. The corresponding general solution is $y_1 = \frac{3}{5}e^{3x} + C_1e^{-x} + C_2e^{-2x}$.

We can then determine y_2 as follows:

$$y_2 = \frac{1}{6}(y_1 - y_1') = \frac{1}{6}(\frac{3}{5}e^{3x} + C_1e^{-x} + C_2e^{-2x} - (\frac{9}{5}e^{3x} - C_1e^{-x} - 2C_2e^{-2x})) = -\frac{1}{5}e^{3x} + \frac{1}{3}C_1e^{-x} + \frac{1}{2}C_2e^{-2x}.$$

Problem 4.

- (a) Write the (third-order) differential equation $y''' + 2y'' - 4y' + 5y = 2\sin(x)$ as a system of (first-order) differential equations.
- (b) Consider the following system of (second-order) initial value problems:

$$\begin{aligned} y_1'' &= 5y_1' + 2y_2' + e^{2x} & y_1(0) &= 1, \quad y_1'(0) = 4, \quad y_2(0) = 0, \quad y_2'(0) = -1 \\ y_2'' &= 7y_1 - 3y_2 - 3e^x \end{aligned}$$

Write it as a first-order initial value problem in the form $\mathbf{y}' = M\mathbf{y} + \mathbf{f}$, $\mathbf{y}(0) = \mathbf{c}$.

Solution.

- (a) Write $y_1 = y$, $y_2 = y'$ and $y_3 = y''$.

Then, the DE translates into the first-order system $\begin{cases} y_1' = y_2 \\ y_2' = y_3 \\ y_3' = -5y_1 + 4y_2 - 2y_3 + 2\sin(x) \end{cases}$.

In matrix form, with $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$, this is $\mathbf{y}' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -5 & 4 & -2 \end{bmatrix} \mathbf{y} + \begin{bmatrix} 0 \\ 0 \\ 2\sin(x) \end{bmatrix}$.

- (b) Introduce $y_3 = y_1'$ and $y_4 = y_2'$. Then, with $\mathbf{y} = (y_1, y_2, y_3, y_4)$, the given system translates into

$$\mathbf{y}' = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 5 & 2 \\ 7 & -3 & 0 & 0 \end{bmatrix} \mathbf{y} + \begin{bmatrix} 0 \\ 0 \\ e^{2x} \\ -3e^x \end{bmatrix}, \quad \mathbf{y}(0) = \begin{bmatrix} 1 \\ 0 \\ 4 \\ -1 \end{bmatrix}.$$

Problem 5. The mixtures in three tanks T_1, T_2, T_3 are kept uniform by stirring. Brine containing 2 lb of salt per gallon enters the first tank at a rate of 15 gal/min. Mixed solution from T_1 is pumped into T_2 at a rate of 10 gal/min and from T_2 into T_3 at a rate of 10 gal/min. Each tank initially contains 10 gal of pure water. Denote by $y_i(t)$ the amount (in pounds) of salt in tank T_i at time t (in minutes). Derive a system of linear differential equations for the y_i , including initial conditions.

Solution. Note that at time t , T_1 contains $10 + 15t - 10t = 10 + 5t$ gal of solution. On the other hand, T_2 contains a constant amount of 10 gal, and T_3 $10 + 10t$ gal of solution.

In the time interval $[t, t + \Delta t]$, we have:

$$\begin{aligned}\Delta y_1 &\approx 15 \cdot 2 \cdot \Delta t - 10 \cdot \frac{y_1}{10 + 5t} \cdot \Delta t &\implies y_1' &= 30 - \frac{2y_1}{2 + t} \\ \Delta y_2 &\approx 10 \cdot \frac{y_1}{10 + 5t} \cdot \Delta t - 10 \cdot \frac{y_2}{10} \cdot \Delta t &\implies y_2' &= \frac{2y_1}{2 + t} - y_2 \\ \Delta y_3 &\approx 10 \cdot \frac{y_2}{10} \cdot \Delta t &\implies y_3' &= y_2\end{aligned}$$

We also have the initial conditions $y_1(0) = 0$, $y_2(0) = 0$, $y_3(0) = 0$. In matrix form, writing $\mathbf{y} = (y_1, y_2, y_3)$, this is

$$\mathbf{y}' = \begin{bmatrix} -\frac{2}{2+t} & 0 & 0 \\ \frac{2}{2+t} & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \mathbf{y} + \begin{bmatrix} 30 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{y}(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

This is a system of linear inhomogeneous differential equations with non-constant coefficients.

Comment. Because of its particularly simple structure, we actually have the techniques to solve this system. Namely, note that the first equation only involves y_1 . It is a linear first-order equation which we could therefore solve using an integrating factor. With y_1 determined, the second differential equation only involves y_2 and is, again, a linear first-order equation. Solving it for y_2 , we then get y_3 by a final integration.

Problem 6.

- What is the period and the amplitude of $3\cos(7t) - 5\sin(7t)$?
- Assume that the angle $\theta(t)$ of a swinging pendulum is described by $\theta'' + 4\theta = 0$. Suppose $\theta(0) = \frac{3}{10}$ and $\theta'(0) = -\frac{4}{5}$. What is the period and the amplitude of the resulting oscillations?
- The position $y(t)$ of a certain mass on a spring is described by $y'' + dy' + 5y = 0$. For which value of $d > 0$ is the system underdamped? Critically damped? Overdamped?
- A forced mechanical oscillator is described by $y'' + 2y' + y = 25\cos(2t)$. As $t \rightarrow \infty$, what is the period and the amplitude of the resulting oscillations?
- The motion of a certain mass on a spring is described by $y'' + y' + \frac{1}{2}y = 5\sin(t)$ with $y(0) = 2$ and $y'(0) = 0$. Determine $y(t)$. As $t \rightarrow \infty$, what are the period and amplitude of the oscillations?

Solution.

- The period is $2\pi/7$ and the amplitude is $\sqrt{3^2 + (-5)^2} = \sqrt{34}$.
- The characteristic equation has roots $\pm 2i$. Hence, the general solution to the DE is $\theta(t) = A\cos(2t) + B\sin(2t)$. We use the initial conditions to determine A and B : $\theta(0) = A = \frac{3}{10}$, $\theta'(0) = 2B = -\frac{4}{5}$. Hence, the unique solution to the IVP is $\theta(t) = \frac{3}{10}\cos(2t) - \frac{2}{5}\sin(2t)$. In particular, the period is π and the amplitude is $\sqrt{A^2 + B^2} = \sqrt{\frac{9}{100} + \frac{16}{100}} = \frac{1}{2}$.

- (c) The characteristic equation has roots $\frac{1}{2}(-d \pm \sqrt{d^2 - 20})$. The system is underdamped if the solutions involve oscillations, which happens if and only if $d^2 - 20$ (the discriminant) is negative.

Since $d^2 - 20 < 0$ if $d < \sqrt{20}$, the system is underdamped for $d \in (0, \sqrt{20})$.

Correspondingly, the system is critically damped for $d = \sqrt{20}$ and overdamped for $d > \sqrt{20}$.

- (d) The characteristic roots of the homogeneous DE are $-1, -1$ while the roots for the inhomogeneous part are $\pm 2i$. Since they don't overlap, there must be a particular solution y_p of the form $y_p = A \cos(2t) + B \sin(2t)$.

We plug into the DE to find $y_p'' + 2y_p' + y_p = (-4A + 4B + A)\cos(2t) + (-4B - 4A + B)\sin(2t) \stackrel{!}{=} 25\cos(2t)$.

Comparing coefficients, we get $-3A + 4B = 25$ and $-3B - 4A = 0$. Solving these, we find $A = -3$ and $B = 4$.

Hence, $y_p(t) = -3\cos(2t) + 4\sin(2t)$ and the general solution is $y(t) = -3\cos(2t) + 4\sin(2t) + (C_1 + C_2x)e^{-t}$.

As $t \rightarrow \infty$, we have $e^{-t} \rightarrow 0$ so that $y(t) \approx -3\cos(2t) + 4\sin(2t)$.

In particular, the period is π and the amplitude is $\sqrt{(-3)^2 + 4^2} = 5$.

- (e) The characteristic roots of the homogeneous DE are $\frac{-2 \pm \sqrt{4-8}}{4} = -\frac{1}{2} \pm \frac{1}{2}i$ while the roots for the inhomogeneous part are $\pm i$. Since there is no overlap, there must be a particular solution y_p of form $y_p = A \cos(t) + B \sin(t)$. By plugging into DE, we find $A = -4$, $B = -2$.

Hence, the general solution is $y(t) = -4\cos(t) - 2\sin(t) + e^{-t/2}(C_1 \cos(t/2) + C_2 \sin(t/2))$.

We determine C_1 and C_2 using the initial conditions. From $y(0) = -4 + C_1 \stackrel{!}{=} 2$, we conclude $C_1 = 6$. We then compute $y'(t) = 4\sin(t) - 2\cos(t) - \frac{1}{2}e^{-t/2}(C_1 \cos(t/2) + C_2 \sin(t/2)) + e^{-t/2}(-\frac{1}{2}C_1 \sin(t/2) + \frac{1}{2}C_2 \cos(t/2))$.

Hence, $y'(0) = -2 - \frac{1}{2}C_1 + \frac{1}{2}C_2 \stackrel{!}{=} 0$, from which we conclude that $C_2 = 10$.

Therefore, the unique solution to the IVP is $y(t) = -4\cos(t) - 2\sin(t) + e^{-t/2}(6\cos(t/2) + 10\sin(t/2))$.

For large t , $y(t) \approx -4\cos(t) - 2\sin(t)$ (since $e^{-t/2} \rightarrow 0$). Hence, the period is 2π and the amplitude is $\sqrt{4^2 + 2^2} = \sqrt{20}$.

Problem 7. The position $y(t)$ of a certain mass on a spring is described by $2y'' + dy' + 3y = F(t)$.

- (a) Assume first that there is no external force, i.e. $F(t) = 0$. For which values of d is the system overdamped?
- (b) Now, $F(t) = \sin(4\omega t)$ and the system is undamped, i.e. $d = 0$. For which values of ω , if any, does resonance occur?
- (c) Now, $F(t) = 5\cos(\omega t) - 2\sin(3\omega t)$ and the system is undamped, i.e. $d = 0$. For which values of ω , if any, does resonance occur?

Solution.

- (a) The discriminant of the characteristic equation is $d^2 - 24$. Hence the system is overdamped if $d^2 - 24 > 0$, that is $d > \sqrt{24} = 2\sqrt{6}$.
- (b) The natural frequency is $\sqrt{\frac{3}{2}}$. Resonance therefore occurs if $4\omega = \sqrt{\frac{3}{2}}$ or, equivalently, $\omega = \frac{1}{4}\sqrt{\frac{3}{2}}$.
- (c) The natural frequency is $\sqrt{\frac{3}{2}}$. The external frequencies are ω and 3ω . Resonance therefore occurs if $\omega = \sqrt{\frac{3}{2}}$ or $3\omega = \sqrt{\frac{3}{2}}$. Equivalently, resonance occurs if $\omega = \sqrt{\frac{3}{2}}$ or $\omega = \frac{1}{3}\sqrt{\frac{3}{2}} = \frac{1}{\sqrt{6}}$.

Problem 8.

- (a) Determine the general solution to $y'' - 4y' + 4y = 3e^{2x}$.

- (b) Determine the general solution to the differential equation $y''' - y = e^x + 7$.
- (c) Determine the general solution $y(x)$ to the differential equation $y^{(4)} + 6y''' + 13y'' = 2$. Express the solution using real numbers only.
- (d) Solve the initial value problem $y'' + 2y' + y = 2e^{2x} + e^{-x}$, $y(0) = -1$, $y'(0) = 2$.

Solution.

- (a) The characteristic equation for the corresponding homogeneous DE has roots 2, 2. The inhomogeneous part (on the right-hand side) solves a DE whose characteristic equation has root 2. Hence, by the method of undetermined coefficients, there must be a particular solution of the form $y_p = Ax^2e^{2x}$.

To determine A , we plug into the DE using $y'_p = 2A(x + x^2)e^{2x}$ and $y''_p = 2A(1 + 4x + 2x^2)e^{2x}$:

$$y''_p - 4y'_p + 4y_p = [2A(1 + 4x + 2x^2) - 8A(x + x^2) + 4Ax^2]e^{2x} = 2Ae^{2x} \stackrel{!}{=} 3e^{2x}. \text{ Hence, } A = \frac{3}{2} \text{ so that } y_p = \frac{3}{2}x^2e^{2x}.$$

Accordingly, the general solution is $y(x) = (C_1 + C_2x + \frac{3}{2}x^2)e^{2x}$.

- (b) Let us first solve the homogeneous equation $y''' - y = 0$. Its characteristic polynomial $D^3 - 1 = (D - 1)(D^2 + D + 1)$ has roots 1 and $-\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$. The inhomogeneous part solves a DE whose characteristic equation has roots 0, 1.

Noting the repetition of the root 1, by the method of undetermined coefficients, there must be a particular solution of the form $y_p = Axex + B$.

$$y'_p = A(x + 1)e^x, \quad y''_p = A(x + 2)e^x, \quad y'''_p = A(x + 3)e^x$$

Plugging into the DE, we get $y'''_p - y_p = 3Ae^x - B \stackrel{!}{=} e^x + 7$. Consequently, $A = \frac{1}{3}$, $B = -7$ so that $y_p = -7 + \frac{1}{3}xex$.

Hence, the general solution is $y(x) = -7 + (C_1 + \frac{1}{3}x)e^x + C_2e^{-x/2}\cos\left(\frac{\sqrt{3}}{2}x\right) + C_3e^{-x/2}\sin\left(\frac{\sqrt{3}}{2}x\right)$.

Comment. On the exam, you won't be asked to factor a polynomial of degree 3 (except possibly trivial cases like $D^5 - D^3 = D^3(D^2 - 1)$). Here, factoring $D^3 - 1$ is not too bad: since 1 is clearly a root, we know that $D^3 - 1 = (D - 1) \cdot p_2(D)$ where $p_2(D)$ is quadratic polynomial which we can find by long division: $p_2(D) = \frac{D^3 - 1}{D - 1}$.

- (c) Since $D^4 + 6D^3 + 13D^2 = D^2(D^2 + 6D + 13)$, the characteristic equation for the corresponding homogeneous DE has roots 0, 0, $-3 \pm 2i$. The inhomogeneous part solves a DE whose characteristic equation has root 0. Hence, by the method of undetermined coefficients, there must be a particular solution of the form $y_p = Ax^2$.

Plugging into the DE, we get $y_p^{(4)} + 6y_p''' + 13y_p'' = 26A \stackrel{!}{=} 2$. Thus $A = \frac{1}{13}$ so that $y_p = \frac{1}{13}x^2$.

Hence, the general solution is $y(x) = \frac{1}{13}x^2 + C_1 + C_2x + C_3e^{-3x}\cos(2x) + C_4e^{-3x}\sin(2x)$.

- (d) The characteristic equation for the associated homogeneous DE has roots $-1, -1$. The inhomogeneous part solves a DE whose characteristic equation has roots $-1, 2$.

Hence, by the method of undetermined coefficients, there must be a particular solution of the form $y_p = Ae^{2x} + Bx^2e^{-x}$. To find A, B we plug into the DE. [...] We find $A = \frac{2}{9}$ and $B = \frac{1}{2}$.

Particular solution: $y_p = \frac{2}{9}e^{2x} + \frac{1}{2}x^2e^{-x}$

General solution: $y = \frac{2}{9}e^{2x} + \frac{1}{2}x^2e^{-x} + C_1e^{-x} + C_2xe^{-x}$

Now, we use the initial values [...], to find $y(x) = \frac{2}{9}e^{2x} + \frac{1}{2}x^2e^{-x} - \frac{11}{9}e^{-x} + \frac{1}{3}xe^{-x}$.

Problem 9.

- (a) Consider the differential equation $x^2y'' - 4xy' + 6y = 0$. Find all solutions of the form $y(x) = x^r$.
- (b) Determine the general solution of $x^2y'' - 4xy' + 6y = x^3$.

Solution.

- (a) Plugging $y(x) = x^r$ into the DE, we get $x^2 r(r-1)x^{r-2} - 4x r x^{r-1} + 6x^r = [r(r-1) - 4r + 6]x^r \stackrel{!}{=} 0$.

Since $r(r-1) - 4r + 6 = (r-2)(r-3)$, we find the solutions x^2 and x^3 . Since this is a second-order equation and our solutions are independent, there can be no further solutions.

- (b) We can find a particular solution to this inhomogeneous DE using the method of variation of parameters/constants. From the first part, we know that the corresponding homogeneous DE has the solutions $y_1 = x^2$, $y_2 = x^3$. The Wronskian of these is $W(x) = y_1 y_2' - y_1' y_2 = x^4$.

Put the DE in the form $y'' - 4x^{-1} y' + 6x^{-2} y = f(x)$ with $f(x) = x$. Then, by the method of variation of parameters, a particular solution is given by

$$y_p = -y_1(x) \int \frac{y_2(x)f(x)}{W(x)} dx + y_2(x) \int \frac{y_1(x)f(x)}{W(x)} dx = -x^2 \int 1 dx + x^3 \int \frac{1}{x} dx = -x^3 + x^3 \ln|x|.$$

Hence, the general solution is $y(x) = C_1 x^2 + (C_2 + \ln|x|)x^3$.