Example 129. Consider the following system of (second-order) initial value problems:

$$\begin{aligned} y_1'' &= 2y_1' - 3y_2' + 7y_2 \\ y_2'' &= 4y_1' + y_2' - 5y_1 \end{aligned} \qquad y_1(0)=2,\ y_1'(0)=3,\ y_2(0)=-1,\ y_2'(0)=1$$

Write it as a first-order initial value problem in the form $\boldsymbol{y}' = M\boldsymbol{y}$, $\boldsymbol{y}(0) = \boldsymbol{y}_0$.

Solution. Introduce $y_3 = y_1'$ and $y_4 = y_2'$. Then, the given system translates into

$$\boldsymbol{y}' = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 7 & 2 & -3 \\ -5 & 0 & 4 & 1 \end{bmatrix} \boldsymbol{y}, \quad \boldsymbol{y}(0) = \begin{bmatrix} 2 \\ -1 \\ 3 \\ 1 \end{bmatrix}.$$

Example 130. Consider the following system of initial value problems:

$$\begin{aligned} y_1''' &= 2y_1'' - 3y_1 + 7y_2 \\ y_2'' &= 4y_1' + y_2' + 5y_2 \end{aligned} \qquad y_1(0)=2,\ y_1'(0)=3,\ y_1''(0)=4,\ y_2(0)=-1,\ y_2'(0)=1$$

Write it as a first-order initial value problem in the form $\boldsymbol{y}' = M\boldsymbol{y}$, $\boldsymbol{y}(0) = \boldsymbol{y}_0$.

Solution. Introduce $y_3 = y_1'$, $y_4 = y_1''$ and $y_5 = y_2'$. Then, the given system translates into

$$\boldsymbol{y}' = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ -3 & 7 & 0 & 2 & 0 \\ 0 & 5 & 4 & 0 & 1 \end{bmatrix} \boldsymbol{y}, \quad \boldsymbol{y}(0) = \begin{bmatrix} 2 \\ -1 \\ 3 \\ 4 \\ 1 \end{bmatrix}.$$

Solving systems of differential equations

Example 131. Determine the general solution to $y_1' = 5y_1 + 4y_2 + e^{2x}$, $y_2' = 8y_1 + y_2$.

Solution. From the second equation it follows that $y_1 = \frac{1}{8}(y_2' - y_2)$. Using this in the first equation, we get $\frac{1}{8}(y_2'' - y_2') = \frac{5}{8}(y_2' - y_2) + 4y_2 + e^{2x}$. After multiplying with 8, this is $y_2'' - y_2' = 5(y_2' - y_2) + 32y_2 + 8e^{2x}$. Simplified, this is $y_2'' - 6y_2' - 27y_2 = 8e^{2x}$, which is an inhomogeneous linear DE with constant coefficients which we know how to solve:

- Since the characteristic roots of the homogeneous DE are $-3, 9$, while the characteristic root for the inhomogeneous part is 2, there must be a particular solution of the form $y_p = Ce^{2x}$. Plugging this y_p into the DE, we get $y_p'' - 6y_p' - 27y_p = (4 - 6\cdot 2 - 27)Ce^{2x} = -35Ce^{2x} \overset{!}{=} 8e^{2x}$. Hence, $C = -\frac{8}{35}$.
- We therefore obtain $y_2 = -\frac{8}{35}e^{2x} + C_1e^{-3x} + C_2e^{9x}$ as the general solution to the inhomogeneous DE.

We can then determine y_1 as

$$\begin{aligned} y_1 &= \frac{1}{8}(y_2' - y_2) \\ &= \frac{1}{8}\left(-\frac{16}{35}e^{2x} - 3C_1e^{-3x} + 9C_2e^{9x} + \frac{8}{35}e^{2x} - C_1e^{-3x} - C_2e^{9x}\right) \\ &= -\frac{1}{35}e^{2x} - \frac{1}{2}C_1e^{-3x} + C_2e^{9x}. \end{aligned}$$

Solution. (alternative) We can also start with $y_2 = \frac{1}{4}y_1' - \frac{5}{4}y_1 - \frac{1}{4}e^{2x}$ (from the first equation), although the algebra will require a little more work. In that case, we have $y_2' = \frac{1}{4}y_1'' - \frac{5}{4}y_1' - \frac{1}{2}e^{2x}$. Using this in the second equation, we get $\frac{1}{4}y_1'' - \frac{5}{4}y_1' - \frac{1}{2}e^{2x} = 8y_1 + \frac{1}{4}y_1' - \frac{5}{4}y_1 - \frac{1}{4}e^{2x}$.

Simplified, this is $y_1'' - 6y_1' - 27y_1 = e^{2x}$, which is an inhomogeneous linear DE with constant coefficients which know how to solve:

- Since the characteristic roots of the homogeneous DE are $-3, 9$, while the characteristic root for the inhomogeneous part is 2, there must be a particular solution of the form $y_p = Ce^{2x}$. Plugging this y_p into the DE, we get $y_p'' - 6y_p' - 27y_p = (4 - 6\cdot 2 - 27)Ce^{2x} = -35Ce^{2x} \overset{!}{=} e^{2x}$. Hence, $C = -\frac{1}{35}$.
- We therefore obtain $y_1 = -\frac{1}{35}e^{2x} + C_1e^{-3x} + C_2e^{9x}$ as the general solution to the inhomogeneous DE.

We can then determine y_2 as

$$\begin{aligned} y_2 &= \frac{1}{4}y_1' - \frac{5}{4}y_1 - \frac{1}{4}e^{2x} \\ &= \frac{1}{4}\left(-\frac{2}{35}e^{2x} - 3C_1e^{-3x} + 9C_2e^{9x}\right) - \frac{5}{4}\left(-\frac{1}{35}e^{2x} + C_1e^{-3x} + C_2e^{9x}\right) - \frac{1}{4}e^{2x} \\ &= -\frac{8}{35}e^{2x} - 2C_1e^{-3x} + C_2e^{9x}. \end{aligned}$$

Important. Make sure you can explain why both of our solutions are equivalent!

Example 132.

(a) Determine the general solution to $y_1' = 5y_1 + 4y_2,\ y_2' = 8y_1 + y_2$.

Comment. In matrix form, with $\boldsymbol{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, this is $\boldsymbol{y}' = \begin{bmatrix} 5 & 4 \\ 8 & 1 \end{bmatrix}\boldsymbol{y}$.

(b) Solve the IVP $y_1' = 5y_1 + 4y_2,\ y_2' = 8y_1 + y_2,\ y_1(0) = 0,\ y_2(0) = 1$.

Solution.

(a) Note that this is the homogeneous system corresponding to the previous problem. It therefore follows from our previous solution (we're using the former one) that $y_1 = -\frac{1}{2}C_1e^{-3x} + C_2e^{9x}$ and $y_2 = C_1e^{-3x} + C_2e^{9x}$ is the general solution of the homogeneous system.

Alternatively. We can, of course, also use the latter solution to write $y_1 = C_1e^{-3x} + C_2e^{9x}$ and $y_2 = -2C_1e^{-3x} + C_2e^{9x}$. These are different ways to write the same general solution.

(b) We already have the general solutions y_1, y_2 to the two DEs. We need to determine the (unique) values of C_1 and C_2 to match the initial conditions: $y_1(0) = -\frac{1}{2}C_1 + C_2 \overset{!}{=} 0,\ y_2(0) = C_1 + C_2 \overset{!}{=} 1$

We solve these two equations and find $C_1 = \frac{2}{3}$ and $C_2 = \frac{1}{3}$.

The unique solution to the IVP therefore is $y_1 = -\frac{1}{3}e^{-3x} + \frac{1}{3}e^{9x}$ and $y_2 = \frac{2}{3}e^{-3x} + \frac{1}{3}e^{9x}$.