Variation of constants for solving inhomogeneous linear DEs

Review. To find the general solution of an inhomogeneous linear DE $Ly = f(x)$, we only need to find a single **particular solution** y_p. Then the general solution is $y_p + y_h$, where y_h is the general solution of $Ly = 0$.

The **method of undetermined coefficients** allows us to find a particular solution to an inhomogeneous linear DE $Ly = f(x)$ for certain functions $f(x)$.

Moreover, the homogeneous DE needs to have constant coefficients.

The next method, known as **variation of constants** (or variation of parameters), has no restriction on the functions $f(x)$ (or the linear DE). The price to pay for this is that the method is usually more laborious.

Theorem 121. (variation of constants) A particular solution to the inhomogeneous second-order linear DE $Ly = y'' + P_1(x)y' + P_0(x)y = f(x)$ is given by:

$$y_p = C_1(x)y_1(x) + C_2(x)y_2(x), \quad C_1(x) = -\int \frac{y_2(x)f(x)}{W(x)}\mathrm{d}x, \quad C_2(x) = \int \frac{y_1(x)f(x)}{W(x)}\mathrm{d}x,$$

where y_1, y_2 are independent solutions of $Ly = 0$ and $W = y_1y_2' - y_1'y_2$ is their Wronskian.

Comment. We obtain the general solution if we consider all possible constants of integration in the formula for y_p.

Proof. Let us look for a particular solution of the form $y_p = C_1(x)\,y_1(x) + C_2(x)\,y_2(x)$.
This "ansatz" is called **variation of constants/parameters**. We plug into the DE to determine conditions on C_1, C_2 so that y_p is a solution. The DE will give us one condition and (since there are two unknowns), it is reasonable to expect that we can impose a second condition (labelled below as "our wish") to make our life simpler.
We compute $y_p' = \underbrace{C_1'y_1 + C_2'y_2}_{=0 \text{ (our wish)}} + C_1y_1' + C_2y_2'$ and, thus, $y_p'' = C_1'y_1' + C_2'y_2' + C_1y_1'' + C_2y_2''$.

["Our wish" was chosen so that y_p'' would only involve first derivatives of C_1 and C_2.]

Therefore, plugging into the DE results in

$$Ly_p = C_1'y_1' + C_2'y_2' + \underbrace{C_1y_1'' + C_2y_2'' + P_1(x)(C_1y_1' + C_2y_2') + P_0(x)(C_1y_1 + C_2y_2)}_{=C_1Ly_1 + C_2Ly_2 = 0} \overset{!}{=} f(x).$$

We conclude that y_p solves the DE if the following two conditions (the first is "our wish") are satisfied:

$$\begin{aligned} C_1'y_1 + C_2'y_2 &= 0, \\ C_1'y_1' + C_2'y_2' &= f(x). \end{aligned}$$

These are linear equations in C_1' and C_2'. Solving gives $C_1' = \frac{-y_2 f(x)}{y_1y_2' - y_1'y_2}$ and $C_2' = \frac{y_1 f(x)}{y_1y_2' - y_1'y_2}$, and it only remains to integrate. □

Comment. In matrix-vector form, the equations are $\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ f(x) \end{bmatrix}$.

Our solution then follows from multiplying $\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}^{-1} = \frac{1}{y_1y_2' - y_1'y_2}\begin{bmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{bmatrix}$ with $\begin{bmatrix} 0 \\ f(x) \end{bmatrix}$.

Advanced comment. $W = y_1y_2' - y_1'y_2$ is called the **Wronskian** of y_1 and y_2. In general, given a linear homogeneous DE of order n with solutions $y_1, \ldots, y_n$, the Wronskian of $y_1, \ldots, y_n$ is the determinant of the matrix where each column consists of the derivatives of one of the y_i. One useful property of the Wronskian is that it is nonzero if and only if the $y_1, \ldots, y_n$ are linearly independent and therefore generate the general solution.

Armin Straub
straub@southalabama.edu

Example 122. Determine the general solution of $y'' - 2y' + y = \frac{e^x}{x}$.

Solution. This DE is of the form $Ly = f(x)$ with $L = D^2 - 2D + 1$ and $f(x) = \frac{e^x}{x}$.

Since $L = (D-1)^2$, the homogeneous DE has the two solutions $y_1 = e^x$, $y_2 = xe^x$.

The corresponding Wronskian is $W = y_1 y_2' - y_1' y_2 = e^x(1+x)e^x - e^x(xe^x) = e^{2x}$.

By variation of parameters (Theorem 121), we find that a particular solution is

$$y_p = -y_1 \int \frac{y_2 f}{W} \mathrm{d}x + y_2 \int \frac{y_1 f}{W} \mathrm{d}x = -e^x \int 1 \mathrm{d}x + xe^x \int \frac{1}{x} \mathrm{d}x = xe^x(\ln|x| - 1).$$

The general solution therefore is $xe^x(\ln|x| - 1) + (C_1 + C_2 x)e^x$.

If we prefer, a simplified particular solution is $xe^x \ln|x|$ (because we can add any multiple of xe^x to y_p). Then the general solution takes the simplified form $xe^x \ln|x| + (C_1 + C_2 x)e^x$.

Comment. Adding constants of integration in the formula for y_p, we get $-e^x(x + D_1) + xe^x(\ln|x| + D_2)$, which is the general solution. Any choice of constants suffices to give us a particular solution.

Important comment. Note that we cannot use the method of undetermined coefficients here because the inhomogeneous term $f(x) = \frac{e^x}{x}$ is not of the appropriate form. See the next example for a case where both methods can be applied.

Example 123. (homework) Determine the general solution of $y'' + 4y' + 4y = e^{3x}$.

(a) Using the method of undetermined coefficients.

(b) Using variation of constants.

Solution.

(a) We already did this in Example 95: The characteristic roots are $-2, -2$. The roots for the inhomogeneous part are 3. Hence, there has to be a particular solution of the form $y_p = Ce^{3x}$. To find the value of C, we plug into the DE.

$y_p'' + 4y_p' + 4y_p = (9 + 4 \cdot 3 + 4)Ce^{3x} \overset{!}{=} e^{3x}$. Hence, $C = 1/25$.

Therefore, the general solution is $y(x) = \frac{1}{25} e^{3x} + (C_1 + C_2 x)e^{-2x}$.

(b) This DE is of the form $Ly = f(x)$ with $L = D^2 + 4D + 4$ and $f(x) = e^{3x}$.

Since $L = (D+2)^2$, the homogeneous DE has the two solutions $y_1 = e^{-2x}$, $y_2 = xe^{-2x}$.

The corresponding Wronskian is $W = y_1 y_2' - y_1' y_2 = e^{-2x}(1-2x)e^{-2x} - (-2e^{-2x})xe^{-2x} = e^{-4x}$.

By variation of parameters (Theorem 121), we find that a particular solution is

$$\begin{aligned} y_p &= -y_1 \int \frac{y_2 f}{W} \mathrm{d}x + y_2 \int \frac{y_1 f}{W} \mathrm{d}x \\ &= -e^{-2x} \underbrace{\int xe^{5x} \mathrm{d}x}_{= \frac{1}{5}xe^{5x} - \frac{1}{25}e^{5x}} + xe^{-2x} \underbrace{\int e^{5x} \mathrm{d}x}_{= \frac{1}{5}e^{5x}} = \frac{1}{25} e^{3x}. \end{aligned}$$

The general solution therefore is $\frac{1}{25} e^{3x} + (C_1 + C_2 x)e^{-2x}$, which matches what we got before.

Example 124. (homework) Determine the general solution of $y'' + 4y' + 4y = 7e^{-2x}$.

(a) Using the method of undetermined coefficients.

(b) Using variation of constants.

Solution.

(a) We already did this in Example 96: The characteristic roots are $-2, -2$. The roots for the inhomogeneous part are -2. Hence, there has to be a particular solution of the form $y_p = Cx^2e^{-2x}$. To find the value of C, we plug into the DE.

$y_p' = C(-2x^2 + 2x)e^{-2x}$

$y_p'' = C(4x^2 - 8x + 2)e^{-2x}$

$y_p'' + 4y_p' + 4y_p = 2Ce^{-2x} \overset{!}{=} 7e^{-2x}$

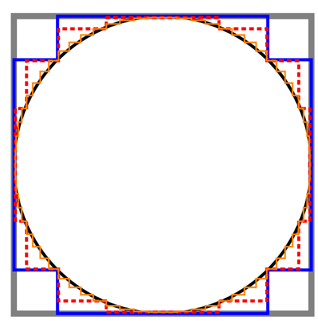
It follows that $C = 7/2$, so that $y_p = \frac{7}{2}x^2e^{-2x}$. The general solution is $y(x) = \left(C_1 + C_2x + \frac{7}{2}x^2\right)e^{-2x}$.

(b) This DE is of the form $Ly = f(x)$ with $L = D^2 + 4D + 4$ and $f(x) = 7e^{-2x}$.
Since $L = (D+2)^2$, the homogeneous DE has the two solutions $y_1 = e^{-2x}$, $y_2 = xe^{-2x}$.
The corresponding Wronskian is $W = y_1y_2' - y_1'y_2 = e^{-2x}(1-2x)e^{-2x} - (-2e^{-2x})xe^{-2x} = e^{-4x}$.
By variation of parameters (Theorem 121), we find that a particular solution is

$$\begin{aligned} y_p &= -y_1\int \frac{y_2 f}{W}\mathrm{d}x + y_2\int \frac{y_1 f}{W}\mathrm{d}x \\ &= -e^{-2x}\underbrace{\int 7x\,\mathrm{d}x}_{=\frac{7}{2}x^2} + xe^{-2x}\underbrace{\int 7\,\mathrm{d}x}_{=7x} = \frac{7}{2}x^2e^{-2x}. \end{aligned}$$

The general solution therefore is $\frac{7}{2}x^2e^{-2x} + (C_1 + C_2x)e^{-2x}$, which matches what we got before.

(Halloween scare!) π is the perimeter of a circle enclosed in a square with edge length 1. The perimeter of the square is 4, which approximates π. To get a better approximation, we "fold" the vertices of the square towards the circle (and get the blue polygon). This construction can be repeated for even better approximations and, in the limit, our shape will converge to the true circle. At each step, the perimeter is 4, so we conclude that $\pi = 4$, contrary to popular belief.



Can you pin-point the fallacy in this argument?

(We are not doing something completely silly! For instance, the areas of our approximations do converge to $\pi/4$, the area of the circle.)

We will talk about a "solution" to the Halloween scare later...

Armin Straub
straub@southalabama.edu