Example 101. (review) What is the shape of a particular solution of $y'' + 4y' + 4y = 4e^{3x}\sin(2x) - x\sin(x)$?

Solution. The characteristic roots are $-2, -2$. The roots for the inhomogeneous part are $3 \pm 2i, \pm i, \pm i$. Hence, there has to be a particular solution of the form
$y_p = A_1 e^{3x}\cos(2x) + A_2 e^{3x}\sin(2x) + (A_3 + A_4 x)\cos(x) + (A_5 + A_6 x)\sin(x)$.

Continuing to find a particular solution. To find the values of $A_1, \ldots, A_6$, we plug into the DE. But this final step is so boring that we don't go through it here. Computers (currently?) cannot afford to be as selective; mine obediently calculated: $y_p = -\frac{4}{841}e^{3x}(20\cos(2x) - 21\sin(2x)) + \frac{1}{125}((-22 + 20x)\cos(x) + (4 - 15x)\sin(x))$

Example 102. What is the shape of a particular solution of $y'' + 4y' + 4y = 4e^{3x}\sin(2x) - x\sin(x) + 5xe^{-2x}$?

Solution. The characteristic roots are $-2, -2$. The roots for the inhomogeneous part are $3 \pm 2i, \pm i, \pm i, -2, -2$. Hence, there has to be a particular solution of the form
$y_p = A_1 e^{3x}\cos(2x) + A_2 e^{3x}\sin(2x) + (A_3 + A_4 x)\cos(x) + (A_5 + A_6 x)\sin(x) + (A_7 x^2 + A_8 x^3)e^{-2x}$.

A closer look at second-order linear DEs

Application: motion of a mass on a spring

Example 103. The motion of a mass m attached to a spring is described by

$$my'' + ky = 0$$

where y is the displacement from the equilibrium position and $k > 0$ is the spring constant.

Why? This follows from Newton's second law $F = ma = my''$ combined with Hooke's law $F = -ky$. (Note that the minus sign is needed because the force on the mass is in direction opposite to the displacement.)

Comment. By measuring y as the displacement from equilibrium, it doesn't matter whether the mass is attached horizontally or vertically (gravity is taken into account by the extra stretch in the spring due to the mass).

Solving this DE, we find that the general solution is

$$y(t) = A\cos(\omega t) + B\sin(\omega t)$$

where $\omega = \sqrt{k/m}$ (note that the characteristic roots are $\pm i\sqrt{\frac{k}{m}}$).

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