Example 96. Determine the general solution of $y'' + 4y' + 4y = 7e^{-2x}$.

Solution. The homogeneous DE is $y'' + 4y' + 4y = 0$ (note that $D^2 + 4D + 4 = (D+2)^2$) and the inhomogeneous part is $7e^{-2x}$.

	homogeneous DE	inhomogeneous part
characteristic roots	$-2, -2$	-2
solutions	e^{-2x}, xe^{-2x}	x^2e^{-2x}

This tells us that there exists a particular solution of the form $y_p = Ax^2 e^{-2x}$. To find the value of A, we plug into the DE.

$y_p' = A(-2x^2 + 2x)e^{-2x}$

$y_p'' = A(4x^2 - 8x + 2)e^{-2x}$

$y_p'' + 4y_p' + 4y_p = 2Ae^{-2x} \overset{!}{=} 7e^{-2x}$

It follows that $A = \frac{7}{2}$, so that $y_p = \frac{7}{2}x^2e^{-2x}$. Hence the general solution is

$$y(x) = \left(C_1 + C_2x + \frac{7}{2}x^2\right)e^{-2x}.$$

Example 97. Consider the DE $y'' + 4y' + 4y = 2e^{3x} - 5e^{-2x}$.

(a) What is the simplest form (with undetermined coefficients) of a particular solution?

(b) Determine a particular solution using our results from Examples 95 and 96.

(c) Determine the general solution.

Solution.

(a) Note that $D^2 + 4D + 4 = (D+2)^2$.

	homogeneous DE	inhomogeneous part
characteristic roots	$-2, -2$	$3, -2$
solutions	e^{-2x}, xe^{-2x}	e^{3x}, x^2e^{-2x}

Hence, there has to be a particular solution of the form $y_p = Ae^{3x} + Bx^2e^{-2x}$.

To find the (unique) values of A and B, we can plug into the DE. Alternatively, we can break the problem into two pieces as illustrated in the next part.

(b) Write the DE as $Ly = 2e^{3x} - 5e^{-2x}$ where $L = D^2 + 4D + 4$. In Example 95 we found that $y_1 = \frac{6}{25}e^{3x}$ satisfies $Ly_1 = 6e^{3x}$. Also, in Example 96 we found that $y_2 = \frac{7}{2}x^2e^{-2x}$ satisfies $Ly_2 = 7e^{-2x}$.

By linearity, it follows that $L(Ay_1 + By_2) = ALy_1 + BLy_2 = 6Ae^{3x} + 7Be^{-2x}$.

To get a particular solution y_p of our DE, we need $6A = 2$ and $7B = -5$.

Hence, $y_p = \frac{2}{6}y_1 - \frac{5}{7}y_2 = \frac{2}{25}e^{3x} - \frac{5}{2}x^2e^{-2x}$.

Comment. Of course, if we hadn't previously solved Examples 95 and 96, we could have plugged the result from the first part into the DE to determine the coefficients A and B. On the other hand, breaking the inhomogeneous part $(2e^{3x} - 5e^{-2x})$ up into pieces (here, e^{3x} and e^{-2x}) can help keep things organized, especially when working by hand.

(c) The general solution is $\frac{2}{25}e^{3x} - \frac{5}{2}x^2e^{-2x} + (C_1 + C_2x)e^{2x}$.

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Example 98. Consider the DE $y'' - 2y' + y = 5\sin(3x)$.

(a) What is the simplest form (with undetermined coefficients) of a particular solution?

(b) Determine a particular solution.

(c) Determine the general solution.

Solution. Note that $D^2 - 2D + 1 = (D-1)^2$.

	homogeneous DE	inhomogeneous part
characteristic roots	$1, 1$	$\pm 3i$
solutions	e^x, xe^x	$\cos(3x), \sin(3x)$

(a) This tells us that there exists a particular solution of the form $y_p = A\cos(3x) + B\sin(3x)$.

(b) To find the values of A and B, we plug into the DE.
$y_p' = -3A\sin(3x) + 3B\cos(3x)$
$y_p'' = -9A\cos(3x) - 9B\sin(3x)$
$y_p'' - 2y_p' + y_p = (-8A - 6B)\cos(3x) + (6A - 8B)\sin(3x) \overset{!}{=} 5\sin(3x)$
Equating the coefficients of $\cos(x)$, $\sin(x)$, we obtain the two equations $-8A - 6B = 0$ and $6A - 8B = 5$.
Solving these, we find $A = \frac{3}{10}$, $B = -\frac{2}{5}$. Accordingly, a particular solution is $y_p = \frac{3}{10}\cos(3x) - \frac{2}{5}\sin(3x)$.

(c) The general solution is $y(x) = \frac{3}{10}\cos(3x) - \frac{2}{5}\sin(3x) + (C_1 + C_2 x)e^x$.

Example 99. Consider the DE $y'' - 2y' + y = 5e^{2x}\sin(3x) + 7xe^x$. What is the simplest form (with undetermined coefficients) of a particular solution?

Solution. Since $D^2 - 2D + 1 = (D-1)^2$, the characteristic roots are $1, 1$. The roots for the inhomogeneous part are $2 \pm 3i, 1, 1$. Hence, there has to be a particular solution of the form $y_p = A_1 e^{2x}\cos(3x) + A_2 e^{2x}\sin(3x) + A_3 x^2 e^x + A_4 x^3 e^x$.
(We can then plug into the DE to determine the (unique) values of the coefficients A_1, A_2, A_3, A_4.)

Example 100. (homework) What is the shape of a particular solution of $y'' + 4y' + 4y = x\cos(x)$?

Solution. The characteristic roots are $-2, -2$. The roots for the inhomogeneous part are $\pm i, \pm i$. Hence, there has to be a particular solution of the form $y_p = (A_1 + A_2 x)\cos(x) + (A_3 + A_4 x)\sin(x)$.

Continuing to find a particular solution. To find the value of the A_j's, we plug into the DE.
$y_p' = (A_2 + A_3 + A_4 x)\cos(x) + (A_4 - A_1 - A_2 x)\sin(x)$
$y_p'' = (2A_4 - A_1 - A_2 x)\cos(x) + (-2A_2 - A_3 - A_4 x)\sin(x)$
$y_p'' + 4y_p' + 4y_p = (3A_1 + 4A_2 + 4A_3 + 2A_4 + (3A_2 + 4A_4)x)\cos(x)$
$\quad + (-4A_1 - 2A_2 + 3A_3 + 4A_4 + (-4A_2 + 3A_4)x)\sin(x) \overset{!}{=} x\cos(x)$.
Equating the coefficients of $\cos(x)$, $x\cos(x)$, $\sin(x)$, $x\sin(x)$, we get the equations $3A_1 + 4A_2 + 4A_3 + 2A_4 = 0$, $3A_2 + 4A_4 = 1$, $-4A_1 - 2A_2 + 3A_3 + 4A_4 = 0$, $-4A_2 + 3A_4 = 0$.
Solving (this is tedious!), we find $A_1 = -\frac{4}{125}$, $A_2 = \frac{3}{25}$, $A_3 = -\frac{22}{125}$, $A_4 = \frac{4}{25}$.
Hence, $y_p = \left(-\frac{4}{125} + \frac{3}{25}x\right)\cos(x) + \left(-\frac{22}{125} + \frac{4}{25}x\right)\sin(x)$.

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