Example 87. Solve the initial value problem $y''' = 4y'' - 4y'$ with $y(0) = 4$, $y'(0) = 0$, $y''(0) = -4$.

Solution. The characteristic polynomial $p(D) = D^3 - 4D^2 + 4D = D(D-2)^2$ has roots $0, 2, 2$.

We conclude that the general solution is $y(x) = C_1 + (C_2 + C_3 x)e^{2x}$.

From this formula for $y(x)$, we compute $y'(x) = (2C_2 + C_3 + 2C_3 x)e^{2x}$ and $y''(x) = 4(C_2 + C_3 + C_3 x)e^{2x}$.
The initial conditions therefore result in the equations $C_1 + C_2 = 4$, $2C_2 + C_3 = 0$, $4C_2 + 4C_3 = -4$.

Solving these (start with the last two equations) we find $C_1 = 3$, $C_2 = 1$, $C_3 = -2$.

Hence the unique solution to the IVP is $y(x) = 3 + (1 - 2x)e^{2x}$.

Review. A linear DE of order n is of the form

$$y^{(n)} + P_{n-1}(x)\, y^{(n-1)} + \ldots + P_1(x) y' + P_0(x) y = f(x).$$

The **general solution of linear DE** always takes the form

$$y(x) = y_p(x) + C_1 y_1(x) + \ldots + C_n\, y_n(x),$$

where y_p is any solution (called a **particular solution**) and $y_1, y_2, \ldots, y_n$ are solutions to the corresponding **homogeneous** linear DE.

- In terms of $D = \frac{\mathrm{d}}{\mathrm{d}x}$, the DE becomes: $Ly = f(x)$ with $L = D^n + P_{n-1}(x) D^{n-1} + \ldots + P_1(x) D + P_0(x)$.
- The inclusion of the $f(x)$ term makes $Ly = f(x)$ an **inhomogeneous** linear DE. The corresponding **homogeneous** DE is $Ly = 0$ (note that the zero function $y(x) = 0$ is a solution of $Ly = 0$).
- L is called a **linear differential operator**.
  - $L(C_1 y_1 + C_1 y_2) = C_1 L y_1 + C_2 L y_2$ **(linearity)**

    Comment. If you are familiar with linear algebra, think of L replaced with a matrix A and y_1, y_2 replaced with vectors $\boldsymbol{v}_1, \boldsymbol{v}_2$. In that case, the same linearity property holds.
  - So, if y_1 solves $Ly = f(x)$, and y_2 solves $Ly = g(x)$, then $C_1 y_1 + C_2 y_2$ solves the differential equation $Ly = C_1 f(x) + C_2\, g(x)$.
  - In particular, if y_1 and y_2 solve the homogeneous DE (then $f(x) = 0$ and $g(x) = 0$), then so does any linear combination $C_1 y_1 + C_2 y_2$. This explains why, for any homogeneous linear DE of order n, there are n solutions $y_1, y_2, \ldots, y_n$ such that the general solution is $y(x) = C_1 y_1(x) + \ldots + C_n\, y_n(x)$. Moreover, in that case, if we have a **particular solution** y_p of the inhomogeneous DE $Ly = f(x)$, then $y_p + C_1 y_1 + \ldots + C_n\, y_n$ is the general solution of $Ly = f(x)$.

Real form of complex solutions

Let's recall some basic facts about **complex numbers**:

- Every complex number can be written as $z = x + iy$ with real x, y.
- Here, the imaginary unit i is characterized by solving $x^2 = -1$.

  Important observation. The same equation is solved by $-i$. This means that, algebraically, we cannot distinguish between $+i$ and $-i$.

- The **conjugate** of $z = x + iy$ is $\bar{z} = x - iy$.

  Important comment. Since we cannot algebraically distinguish between $\pm i$, we also cannot distinguish between z and $\bar{z}$. That's the reason why, in problems involving only real numbers, if a complex number $z = x + iy$ shows up, then its **conjugate** $\bar{z} = x - iy$ has to show up in the same manner. With that in mind, have another look at the examples below.

- The **real part** of $z = x + iy$ is x and we write $\mathrm{Re}(z) = x$.

  Likewise the **imaginary part** is $\mathrm{Im}(z) = y$.

  Observe that $\mathrm{Re}(z) = \frac{1}{2}(z + \bar{z})$ as well as $\mathrm{Im}(z) = \frac{1}{2i}(z - \bar{z})$.

Theorem 88. (Euler's identity) $e^{ix} = \cos(x) + i\sin(x)$

Proof. Observe that both sides are the (unique) solution to the IVP $y' = iy$, $y(0) = 1$.
[Check that by computing the derivatives and verifying the initial condition! As we did in class.] □

On lots of T-shirts. In particular, with $x = \pi$, we get $\boxed{e^{\pi i} = -1}$ or $e^{i\pi} + 1 = 0$ (which connects the five fundamental constants).

Comment. It follows that $\cos(x) = \mathrm{Re}(e^{ix}) = \frac{1}{2}(e^{ix} + e^{-ix})$ and $\sin(x) = \mathrm{Im}(e^{ix}) = \frac{1}{2i}(e^{ix} - e^{-ix})$.

If $r = a \pm bi$ are characteristic roots of a homogeneous linear DE with constant coefficients, then the corresponding solutions are $e^{ax}\cos(bx)$ and $e^{ax}\sin(bx)$.

Why? $e^{(a \pm bi)x} = e^{ax}e^{\pm bix} = e^{ax}(\cos(bx) \pm i\sin(bx))$

Note that if we, for instance, add $e^{(a+bi)x} + e^{(a-bi)x}$ (that's twice the real part), this will give us $2e^{ax}\cos(bx)$.

Example 89. Determine the general solution of $y'' + y = 0$.

Solution. (complex numbers in general solution) The characteristic polynomial is $D^2 + 1$ which has no roots over the reals. Over the **complex numbers**, by definition, the roots are i and $-i$.

So the general solution is $y(x) = C_1 e^{ix} + C_2 e^{-ix}$.

Solution. (real general solution) On the other hand, we easily check that $y_1 = \cos(x)$ and $y_2 = \sin(x)$ are two solutions. Hence, the general solution can also be written as $y(x) = D_1 \cos(x) + D_2 \sin(x)$.

Important comment. That we have these two different representations is a consequence of Euler's identity (Theorem 88) by which $e^{\pm ix} = \cos(x) \pm i\sin(x)$.

On the other hand, $\cos(x) = \frac{1}{2}(e^{ix} + e^{-ix})$ and $\sin(x) = \frac{1}{2i}(e^{ix} - e^{-ix})$.

[Recall that the first formula is an instance of $\mathrm{Re}(z) = \frac{1}{2}(z + \bar{z})$ and the second of $\mathrm{Im}(z) = \frac{1}{2i}(z - \bar{z})$.]