Review. A homogeneous linear DE with constant coefficients is of the form $p(D)y=0$, where $p(D)$ is the characteristic polynomial polynomial. For each characteristic root r of multiplicity k, we get the k solutions $x^j e^{rx}$ for $j=0,1,\ldots,k-1$.

Example 80. (review) Find the general solution of $y'''+2y''+y'=0$.

Solution. The characteristic polynomial $p(D)=D(D+1)^2$ has roots $0,1,1$.
Hence, the general solution is $A+(B+Cx)e^x$.

Example 81. Determine the general solution of $y'''-3y''+3y'-y=0$.

Solution. The characteristic polynomial $p(D)=D^3-3D^2+3D-1=(D-1)^3$ has roots $1,1,1$.
By Theorem 77, the general solution is $y(x)=(C_1+C_2x+C_3x^2)e^x$.

Comment. The coefficients $1,2,1$ and $1,3,3,1$ in $(D+1)^2$ and $(D+1)^3$ are known as binomial coefficients. They can be arranged as rows in Pascal's triangle where the next row would be $1,4,6,4,1$.

Example 82. Determine the general solution of $y'''-y''-5y'-3y=0$.

Solution. The characteristic polynomial $p(D)=D^3-D^2-5D-3=(D-3)(D+1)^2$ has roots $3,-1,-1$.
Hence, the general solution is $y(x)=C_1e^{3x}+(C_2+C_3x)e^{-x}$.

Example 83. Determine the general solution of $y^{(6)}=3y^{(5)}-4y'''$.

Solution. This DE is of the form $p(D)\,y=0$ with $p(D)=D^6-3D^5+4D^3=D^3(D-2)^2(D+1)$.
The characteristic roots are $2,2,0,0,0,-1$.
By Theorem 77, the general solution is $y(x)=(C_1+C_2x)e^{2x}+C_3+C_4x+C_5x^2+C_6e^{-x}$.

Example 84. Consider the function $y(x)=3xe^{-2x}+7e^x$. Determine a homogeneous linear DE with constant coefficients of which $y(x)$ is a solution.

Solution. In order for $y(x)$ to be a solution of $p(D)y=0$, the characteristic roots must include $-2,-2,1$.
The simplest choice for $p(D)$ thus is $p(D)=(D+2)^2(D-1)$.

Note. For many purposes it is best to leave the DE as $(D+2)^2(D-1)y=0$. On the other hand, if we wanted to, we could multiply out $(D+2)^2(D-1)=D^3+3D^2-4$ to write the DE in the more classical form $y'''+3y''-4y=0$.

Example 85. Consider the function $y(x)=3xe^{-2x}+7x^3$. Determine a homogeneous linear DE with constant coefficients of which $y(x)$ is a solution.

Solution. In order for $y(x)$ to be a solution of $p(D)y=0$, the characteristic roots must include $-2,-2,0,0,0,0$.
The simplest choice for $p(D)$ thus is $p(D)=(D+2)^2D^4$.

Note. Optionally, we can expand $(D+2)^2D^4=D^6+4D^5+4D^4$ to write the DE as $y^{(6)}+4y^{(5)}+4y^{(4)}=0$.

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The following gives a preview of how we are going to solve **inhomogeneous** linear differential equations.

Example 86. (preview) Determine the general solution of $y'' + 4y' + 4y = 5$.

Hint: Look for a constant solution.

Solution. We know how to solve the corresponding homogeneous DE: $y'' + 4y' + 4y = 0$. Namely, since the characteristic polynomial is $p(D) = D^2 + 4D + 4 = (D+2)^2$ we get that e^{-2x} and xe^{-2x} solve the homogeneous DE. It follows that our original inhomogeneous DE has the general solution

$$y(x) = y_p(x) + C_1 e^{-2x} + C_2 x e^{-2x}$$

where $y_p(x)$ is a particular solution that we still need to find.
Because of the hint, we look for a particular solution of the form $y_p = A$ where A is a constant. If we plug this into the (original) DE we get $0 + 4 \cdot 0 + 4 \cdot A \overset{!}{=} 5$. This means that we indeed get a solution if $A = \frac{5}{4}$. Hence, the general solution of the original DE is

$$y(x) = \frac{5}{4} + C_1 e^{-2x} + C_2 x e^{-2x}.$$

Just to make sure. The DE in operator notation is $Ly = f(x)$ with $L = D^2 + 4D + 4$ and $f(x) = 5$.
Next. How to find such particular solutions in all cases (and without hints).

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