Linear DEs of higher order

The most general linear first-order DE is of the form $A(x)y' + B(x)y + C(x) = 0$. Any such DE can be rewritten in the form $y' + P(x)y = f(x)$ by dividing by $A(x)$ and rearranging.

We have learned how to solve all of these using an integrating factor.

Linear DEs of order n are those that can be written in the form

$$y^{(n)} + P_{n-1}(x)\, y^{(n-1)} + \ldots + P_1(x)y' + P_0(x)y = f(x).$$

The corresponding **homogeneous linear DE** is the DE

$$y^{(n)} + P_{n-1}(x)\, y^{(n-1)} + \ldots + P_1(x)y' + P_0(x)y = 0,$$

and it plays an important role in solving the original linear DE.

Important note. A linear DE is **homogeneous** if and only if the zero function $y(x) = 0$ is a solution.

Advanced comment. As we observed in the first-order case, if I is an interval on which all the $P_j(x)$ as well as $f(x)$ are continuous, then for any $a \in I$ the IVP with $y(a) = b_0$, $y'(a) = b_1$, ..., $y^{(n-1)}(a) = b_{n-1}$ always has a unique solution (which is defined on all of I).

Theorem 67. (general solution of linear DEs) For a linear DE of order n, the general solution always takes the form

$$y(x) = y_p(x) + C_1 y_1(x) + \ldots + C_n\, y_n(x),$$

where y_p is any single solution (called a **particular solution**) and $y_1, y_2, \ldots, y_n$ are solutions to the corresponding **homogeneous** linear DE.

Comment. If the linear DE is already homogeneous, then the zero function $y(x) = 0$ is a solution and we can use $y_p = 0$. In that case, the general solution is of the form $y(x) = C_1 y_1 + C_2 y_2 + \cdots + C_n y_n$.

Why? This structure of the solution follows from the observation in the next example.

Example 68. Suppose that y_1 solves $y'' + P(x)y' + Q(x)y = f(x)$ and that y_2 solves $y'' + P(x)y' + Q(x)y = g(x)$ (note that the corresponding homogeneous DE is the same).
Show that $7y_1 + 4y_2$ solves $y'' + P(x)y' + Q(x)y = 7f(x) + 4g(x)$.

Solution. $(7y_1 + 4y_2)'' + P(x)(7y_1 + 4y_2)' + Q(x)(7y_1 + 4y_2)$
$= 7\{y_1'' + P(x)y_1' + Q(x)y_1\} + 4\{y_2'' + P(x)y_2' + Q(x)y_2\} = 7 \cdot f(x) + 4 \cdot g(x)$

Comment. Of course, there is nothing special about the coefficients 7 and 4.

Important comment. In particular, if both $f(x)$ and $g(x)$ are zero, then $7f(x) + 4g(x)$ is zero as well. This shows that homogeneous linear DEs have the important property that, if y_1 and y_2 are two solutions, then any linear combination $C_1\, y_1 + C_2\, y_2$ is a solution as well.

The upshot is that this observation reduces the task of finding the general solution of a homogeneous linear DE to the task of finding n (sufficiently) different solutions.

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Example 69. (extra) The DE $x^2y''+2xy'-6y=0$ has solutions $y_1=x^2$, $y_2=x^{-3}$.

(a) Determine the general solution.

(b) Solve the IVP $x^2y''+2xy'-6y=0$ with $y(2)=10$, $y'(2)=15$.

Solution.

(a) Note that this is a homogeneous linear DE of order 2.
Hence, given the two solutions, we conclude that the general solution is $y(x)=Ax^2+Bx^{-3}$ (in this case, the particular solution is $y_p=0$ because the DE is homogeneous).

(b) We already know that the general solution of the DE is $y(x)=Ax^2+Bx^{-3}$.
It follows that $y'(x)=2Ax-3Bx^{-4}$.
We now use the two initial conditions to solve for A and B:
Solving $y(2)=4A+B/8\overset{!}{=}10$ and $y'(2)=4A-3/16B\overset{!}{=}15$ for A and B results in $A=3$, $B=-16$.
Hence, the unique solution to the IVP is $y(x)=3x^2-16/x^3$.

Review. We saw in Theorem 67 that, for a linear DE of order n, the general solution is of the form

$$y(x)=y_p(x)+C_1y_1(x)+\ldots+C_n y_n(x),$$

where y_p is any single solution (called a **particular solution**) and $y_1, y_2, \ldots, y_n$ are solutions to the corresponding **homogeneous** linear DE.

Linear differential equations with constant coefficients

Let us have another look at Example 10. Note that the DE is a second-order linear differential equation with constant coefficients. Our upcoming goal will be to solve all such equations.

Example 70. (warmup) Find the general solution to $y''=y'+6y$.

Solution. As in Example 10, we look for solutions of the form e^{rx}.
Plugging e^{rx} into the DE, we get $r^2e^{rx}=re^{rx}+6e^{rx}$ which simplifies to $r^2-r-6=0$.
This is called the **characteristic equation**. Its solutions are $r=-2,3$ (the **characteristic roots**).
This means we found the two solutions $y_1=e^{-2x}$, $y_2=e^{3x}$.
Since this a homogeneous linear DE (see Theorem 67), the general solution is $y=C_1e^{2x}+C_2e^{-x}$.

Homogeneous linear DEs with constant coefficients

Let us look at another example like Example 70. This time we also take an operator approach that explains further what is going on (and that will be particularly useful when we discuss inhomogeneous equations).

An **operator** takes a function as input and returns a function as output. That is exactly what the derivative does.

In the sequel, we write $D=\frac{\mathrm{d}}{\mathrm{d}x}$ for the derivative operator.

For instance. We write $y'=\frac{\mathrm{d}}{\mathrm{d}x}y=Dy$ as well as $y''=\frac{\mathrm{d}^2}{\mathrm{d}x^2}y=D^2y$.

Example 71. Find the general solution to $y'' - y' - 2y = 0$.

Solution. (our earlier approach) Let us look for solutions of the form e^{rx}.
Plugging e^{rx} into the DE, we get $r^2 e^{rx} - r e^{rx} - 2e^{rx} = 0$.
Equivalently, $r^2 - r - 2 = 0$. This is the characteristic equation. Its solutions are $r = 2, -1$.
This means we found the two solutions $y_1 = e^{2x}$, $y_2 = e^{-x}$.
Since this a homogeneous linear DE (see Theorem 67), the general solution is $y = C_1 e^{2x} + C_2 e^{-x}$.

Solution. (operator approach) $y'' - y' - 2y = 0$ is equivalent to $(D^2 - D - 2)y = 0$.
Note that $D^2 - D - 2 = (D-2)(D+1)$ is the **characteristic polynomial**.
Observe that we get solutions to $(D-2)(D+1)y = 0$ from $(D-2)y = 0$ and $(D+1)y = 0$.
$(D-2)y = 0$ is solved by $y_1 = e^{2x}$, and $(D+1)y = 0$ is solved by $y_2 = e^{-x}$; as in the previous solution.
Again, we conclude that the general solution is $y = C_1 e^{2x} + C_2 e^{-x}$.

Set $D = \frac{\mathrm{d}}{\mathrm{d}x}$. Every **homogeneous linear DE with constant coefficients** can be written as $p(D)y = 0$, where $p(D)$ is a polynomial in D, called the **characteristic polynomial**.

For instance. $y'' - y' - 2y = 0$ is equivalent to $Ly = 0$ with $L = D^2 - D - 2$.

Example 72. Solve $y'' - y' - 2y = 0$ with initial conditions $y(0) = 4$, $y'(0) = 5$.

Solution. From Example 71, we know that the general solution is $y(x) = C_1 e^{2x} + C_2 e^{-x}$.
It follows that $y'(x) = 2C_1 e^{2x} - C_2 e^{-x}$. We now use the two initial conditions to solve for C_1 and C_2:
The initial conditions result in the two equations $y(0) = C_1 + C_2 \overset{!}{=} 4$, $y'(0) = 2C_1 - C_2 \overset{!}{=} 5$.
Solving these we find $C_1 = 3$, $C_2 = 1$.
Hence the unique solution to the IVP is $y(x) = 3e^{2x} + e^{-x}$.

Example 73. (preview of inhomogeneous linear DEs)

(a) Check that $y = -3x$ is a solution to $y'' - y' - 2y = 6x + 3$.

Comment. We will soon learn how to find such a solution from scratch.

(b) Using the first part, determine the general solution to $y'' - y' - 2y = 6x + 3$.

(c) Determine $f(x)$ so that $y = 7x^2$ solves $y'' - y' - 2y = f(x)$.

Comment. This is how you can create problems like the ones in the first two parts.

Solution.

(a) If $y = -3x$, then $y' = -3$ and $y'' = 0$. Plugging into the DE, we find $0 - (-3) - 2 \cdot (-3x) = 6x + 3$, which verifies that this is a solution.

(b) This is an inhomogeneous linear DE. From Example 71, we know that the corresponding homogeneous DE has the general solution $C_1 e^{2x} + C_2 e^{-x}$.
From the first part, we know that $-3x$ is a particular solution.
Combining this, the general solution to the present DE is $-3x + C_1 e^{2x} + C_2 e^{-x}$ (see Theorem 67).

(c) If $y = 7x^2$, then $y' = 14x$ and $y'' = 14$ so that $y'' - y' - 2y = 14 - 14x - 14x^2$.
Thus $f(x) = 14 - 14x - 14x^2$.

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