Midterm #1

MATH 238 — Differential Equations I
Wednesday, Oct 2, 2024

Please print your name:

No notes, calculators or tools of any kind are permitted. There are 30 points in total. You need to show work to receive full credit.

Good luck!

Problem 1. (8 points) A tank holds 10gal of brine containing 40lb of salt. It is filled with brine (containing 5lb/gal salt) at a rate of 3gal/min. At the same time, well-mixed solution flows out at a rate of 2gal/min. How much salt is in the tank after t minutes?

Solution. Let $x(t)$ denote the amount of salt (in lb) in the tank after time t (in min).

At time t, the concentration of salt (in lb/gal) in the tank is $\frac{x(t)}{V(t)}$ where $V(t) = 10 + (3-2)t = 10 + t$ is the volume (in gal) in the tank.

In the time interval $[t, t + \Delta t]$: $\Delta x \approx 3 \cdot 5 \cdot \Delta t - 2 \cdot \frac{x(t)}{V(t)} \cdot \Delta t$.

Hence, $x(t)$ solves the IVP $\frac{\mathrm{d}x}{\mathrm{d}t} = 15 - \frac{2}{10+t} x$ with $x(0) = 40$. Since this is a linear DE, we can solve it as follows:

- We write it in the form $\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{2}{10+t} x = 15$.
- The integrating factor is $f(t) = \exp\left(\int \frac{2}{10+t} \mathrm{d}t\right) = \exp(2\ln(10+t)) = (10+t)^2$.
- Multiply the (rewritten) DE by $f(t) = (10+t)^2$ to get $\underbrace{(10+t)^2 \frac{\mathrm{d}x}{\mathrm{d}t} + 2(10+t)x}_{= \frac{\mathrm{d}}{\mathrm{d}t}[(10+t)^2 x]} = 15(10+t)^2$.
- Integrate both sides to get $(10+t)^2 x = 15 \int (10+t)^2 \mathrm{d}t = 5(10+t)^3 + C$.

Hence the general solution to the DE is $x(t) = \frac{5(10+t)^3 + C}{(10+t)^2}$. Using $x(0) = 40$, we find $40 = \frac{5000 + C}{100}$ from which we conclude that $C = -1000$.

After t minutes, the tank therefore contains $x(t) = \frac{5(10+t)^3 - 1000}{(10+t)^2}$ pounds of salt.

(Depending on preference, we can also write $\frac{5(10+t)^3 - 1000}{(10+t)^2} = 5(10+t) - \frac{1000}{(10+t)^2}$.)

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Problem 2. (3 points) In the differential equation $(x+2y)\frac{\mathrm{d}y}{\mathrm{d}x} = \tan\left(-\frac{y}{x^2}\right)$ substitute $u = \frac{y}{x^2}$.

What is the resulting differential equation for u? No need to simplify! Do not attempt to solve!

Solution. If $u = \frac{y}{x^2}$, then $y = ux^2$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = x^2\frac{\mathrm{d}u}{\mathrm{d}x} + 2ux$.

Hence, the resulting differential equation for u is $(x+2ux^2)\left(x^2\frac{\mathrm{d}u}{\mathrm{d}x} + 2ux\right) = \tan(-u)$.

Problem 3. (3 points) Find the general solution to the differential equation $y'' + y' = 2y$.

Solution. We look for solutions of the form e^{rx}.

Plugging e^{rx} into the DE, we get $r^2e^{rx} + re^{rx} = 2e^{rx}$ which simplifies to $r^2 + r - 2 = 0$.

$r^2 + r - 2 = 0$ has the two solutions $r = \frac{-1 \pm \sqrt{1 - 4\cdot(-2)}}{2} = \frac{-1 \pm 3}{2} = 1, -2$.

This means we found the two solutions $y_1 = e^x$, $y_2 = e^{-2x}$.

The general solution to the DE is $Ae^x + Be^{-2x}$.

Problem 4. (3 points) Consider the initial value problem $(y^2-1)y' + \sin(x) = x^2$, $y(a) = b$. For which values of a and b can we guarantee existence and uniqueness of a (local) solution?

Solution. Let us write $y' = f(x,y)$ with $f(x,y) = \frac{x^2 - \sin(x)}{y^2 - 1}$. Then $\frac{\partial}{\partial y}f(x,y) = -\frac{x^2 - \sin(x)}{(y^2-1)^2}\cdot 2y$.

Both $f(x,y)$ and $\frac{\partial}{\partial y}f(x,y)$ are continuous for all (x,y) with $y \neq \pm 1$.

Hence, if $b \neq \pm 1$, then the IVP locally has a unique solution by the existence and uniqueness theorem.

Problem 5. (3 points) A rising population is modeled by the equation $\frac{\mathrm{d}P}{\mathrm{d}t} = 300P - 3P^2$. Answer the following questions without solving the differential equation.

(a) When the population size stabilizes in the long term, how big will the population be?

(b) What is the population size when it is growing the fastest?

Solution.

(a) Once the population reaches a stable level in the long term, we have $\frac{\mathrm{d}P}{\mathrm{d}t} = 0$ (no change in population size).

Hence, $0 = 300P - 3P^2 = 3P(100 - P)$ which implies that $P = 0$ or $P = 100$. Since the population is rising, it will approach 100 in the long term.

(b) This is asking when $\frac{\mathrm{d}P}{\mathrm{d}t}$ (the population growth) is maximal.

The DE is telling us that this growth is $f(P) = 300P - 3P^2$. This a parabola that opens to the bottom. From Calculus, we know that it has a global maximum when $f'(P) = 0$.

$f'(P) = 300 - 6P = 0$ leads to $P = 50$.

Thus, the population is growing the fastest when its size is 50.

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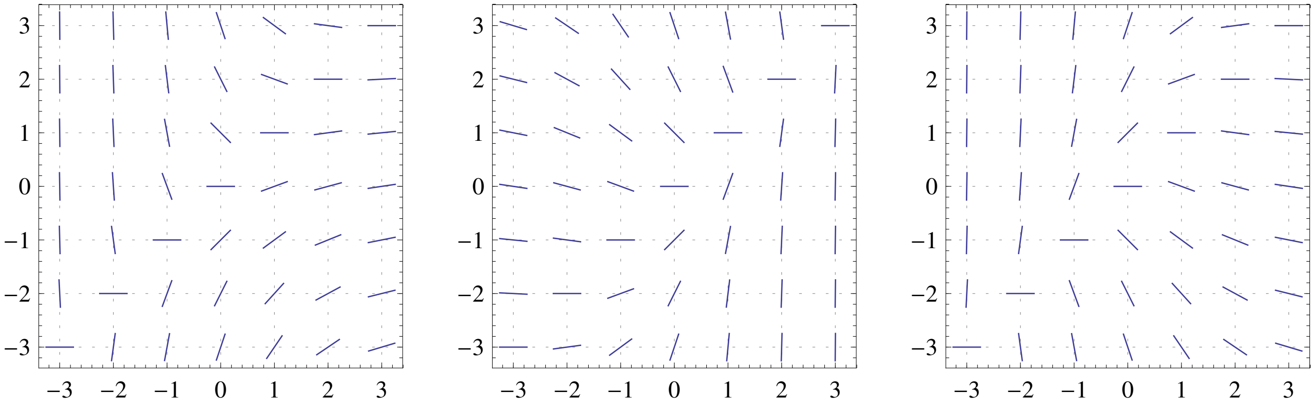
Problem 6. (2 points) Circle the slope field below which belongs to the differential equation $e^x y' = y - x$.



Solution. A good point to carefully consider is $(1, 3)$. By the DE, a solution passing through that point has slope y' satisfying $e^1 y' = 3 - 1$. Equivalently, $y' = 2/e > 0$. The only plot compatible with that is the last one.

Of course, we can arrive at the same conclusion based on other points.

Problem 7. (4 points) Solve the initial value problem $\frac{\mathrm{d}y}{\mathrm{d}x} + y^2 \sin(x) = 0$ with $y(0) = 3$.

Solution. This DE is separable:

$$\frac{\mathrm{d}y}{y^2} = -\sin(x)\,\mathrm{d}x \quad \Longrightarrow \quad \frac{-1}{y} = \cos(x) + C \quad \Longrightarrow \quad y = \frac{-1}{\cos(x) + C}.$$

Using $y(0) = 3$ in $\frac{-1}{y} = \cos(x) + C$, we get $-\frac{1}{3} = 1 + C$ from which we conclude that $C = -\frac{4}{3}$.

Thus, the unique solution is $y = \frac{-1}{\cos(x) - \frac{4}{3}} = \frac{3}{4 - 3\cos(x)}$.

Problem 8. (4 points) Consider the IVP $\frac{\mathrm{d}y}{\mathrm{d}x} - y^2 = x$ with $y(1) = -1$. Approximate the solution $y(x)$ for $x \in [1, 2]$ using Euler's method with 2 steps. In particular, what is the approximation of $y(2)$?

Solution. The step size is $h = \frac{2-1}{2} = \frac{1}{2}$. We to put the DE into the form $y' = f(x, y)$ by rewriting it as $\frac{\mathrm{d}y}{\mathrm{d}x} = x + y^2$. We therefore apply Euler's method with $f(x, y) = x + y^2$:

$$\begin{array}{ll} x_0 = 1 & y_0 = -1 \\ x_1 = \dfrac{3}{2} & y_1 = y_0 + h f(x_0, y_0) = -1 + \dfrac{1}{2} \cdot [1 + (-1)^2] = 0 \\ x_2 = 2 & y_2 = y_1 + h f(x_1, y_1) = 0 + \dfrac{1}{2} \cdot \left[\dfrac{3}{2} + 0^2\right] = \dfrac{3}{4} \end{array}$$

In particular, the approximation for $y(2)$ is $y_2 = \frac{3}{4}$.

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(extra scratch paper)