Solving systems of DEs using Laplace transforms

We solved the following system in Example 125 using elimination and our method for solving linear DEs with constant coefficients based on characteristic roots.

Example 163. Solve the system $y_1' = 5y_1 + 4y_2$, $y_2' = 8y_1 + y_2$, $y_1(0) = 0$, $y_2(0) = 1$.

Solution. (using Laplace transforms) $y_1' = 5y_1 + 4y_2$ transforms into $sY_1 - \underbrace{y_1(0)}_{=0} = 5Y_1 + 4Y_2$.

Likewise, $y_2' = 8y_1 + y_2$ transforms into $sY_2 - \underbrace{y_2(0)}_{=1} = 8Y_1 + Y_2$.

The transformed equations are regular equations that we can solve for Y_1 and Y_2.

For instance, by the first equation, $Y_2 = \frac{1}{4}(s-5)Y_1$.

Used in the second equation, we get $\underbrace{-8Y_1 + \frac{1}{4}(s-1)(s-5)Y_1}_{=\frac{1}{4}(s^2-6s-27)=\frac{1}{4}(s+3)(s-9)} = 1$ so that $Y_1 = \frac{4}{(s+3)(s-9)}$.

Hence, the system is solved by $Y_1 = \frac{4}{(s+3)(s-9)}$ and $Y_2 = \frac{1}{4}(s-5)Y_1 = \frac{s-5}{(s+3)(s-9)}$.

As a final step, we need to take the inverse Laplace transform to get $y_1(t) = \mathcal{L}^{-1}(Y_1(s))$ and $y_2(t) = \mathcal{L}^{-1}(Y_2(s))$.

Using partial fractions, $Y_1(s) = \frac{4}{(s+3)(s-9)} = -\frac{1}{3}\cdot\frac{1}{s+3} + \frac{1}{3}\cdot\frac{1}{s-9}$ so that $y_1(t) = -\frac{1}{3}e^{-3t} + \frac{1}{3}e^{9t}$.

Similarly, $Y_2(s) = \frac{s-5}{(s+3)(s-9)} = \frac{2}{3}\cdot\frac{1}{s+3} + \frac{1}{3}\cdot\frac{1}{s-9}$ so that $y_2(t) = \frac{2}{3}e^{-3t} + \frac{1}{3}e^{9t}$.

Solution. (old solution, for comparison) Since $y_2 = \frac{1}{4}y_1' - \frac{5}{4}y_1$ (from the first eq.), we have $y_2' = \frac{1}{4}y_1'' - \frac{5}{4}y_1'$.

Using these in the second equation, we get $\frac{1}{4}y_1'' - \frac{5}{4}y_1' = 8y_1 + \frac{1}{4}y_1' - \frac{5}{4}y_1$.
Simplified, this is $y_1'' - 6y_1' - 27y_1 = 0$.
This is a homogeneous linear DE with constant coefficients. The characteristic roots are $-3, 9$.
We therefore obtain $y_1 = C_1e^{-3t} + C_2e^{9t}$ as the general solution.
Thus, $y_2 = \frac{1}{4}y_1' - \frac{5}{4}y_1 = \frac{1}{4}(-3C_1e^{-3t} + 9C_2e^{9t}) - \frac{5}{4}(C_1e^{-3t} + C_2e^{9t}) = -2C_1e^{-3t} + C_2e^{9t}$.
We determine the (unique) values of C_1 and C_2 using the initial conditions:

$y_1(0) = C_1 + C_2 \overset{!}{=} 0$
$y_2(0) = -2C_1 + C_2 \overset{!}{=} 1$
We solve these two equations and find $C_1 = -\frac{1}{3}$ and $C_2 = \frac{1}{3}$.

The unique solution to the IVP therefore is $y_1(t) = -\frac{1}{3}e^{-3t} + \frac{1}{3}e^{9t}$ and $y_2(t) = \frac{2}{3}e^{-3t} + \frac{1}{3}e^{9t}$.

Armin Straub
straub@southalabama.edu

Application to military strategy: Lanchester's equations

In military strategy, Lanchester's equations can be used to model two opposing forces during "aimed fire" battle.

Let $x(t)$ and $y(t)$ describe the number of troops on each side. Then Lanchester (during World War I) assumed that the rates $-x'(t)$ and $-y'(t)$, at which soldiers are put out of action, are proportional to the number of opposing forces. That is:

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} -\beta y(t) \\ -\alpha x(t) \end{bmatrix}, \quad \text{or, in matrix form: } \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & -\beta \\ -\alpha & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

The proportionality constants $\alpha, \beta > 0$ indicate the strength of the forces ("fighting effectiveness coefficients").

These are simple linear DEs with constant coefficients, which we have learned how to solve.

Comment. The "aimed fire" means that all combatants are engaged, as is common in modern combat with long-range weapons. This is rather different than ancient combat where only some of the soldiers (such as those in front) were engaged at a time. For more details, see: `https://en.wikipedia.org/wiki/Lanchester%27s_laws`

Example 164. Solve Lanchester's equations with initial conditions $x(0) = x_0$ and $y(0) = y_0$.

Solution. (using Laplace transforms) $x' = -\beta y$ transforms into $sX - x_0 = -\beta Y$. Likewise, $y' = -\alpha x$ transforms into $sY - y_0 = -\alpha X$. The transformed equations are regular equations that we can solve for X and Y. For instance, by the first equation, $Y = -\frac{1}{\beta}(sX - x_0)$.

Used in the second equation, we get $-\frac{s}{\beta}(sX - x_0) - y_0 = -\alpha X$ so that $(s^2 - \alpha\beta)X = sx_0 - \beta y_0$.

Hence, the system is solved by $X = \frac{sx_0 - \beta y_0}{s^2 - \alpha\beta}$ and $Y = -\frac{1}{\beta}(sX - x_0) = \frac{sy_0 - \alpha x_0}{s^2 - \alpha\beta}$.

As a final step, we need to take the inverse Laplace transform to get $x(t) = \mathcal{L}^{-1}(X(s))$ and $y(t) = \mathcal{L}^{-1}(Y(s))$.

Using partial fractions, $X(s) = \frac{sx_0 - \beta y_0}{(s - \sqrt{\alpha\beta})(s + \sqrt{\alpha\beta})} = \frac{A}{s - \sqrt{\alpha\beta}} + \frac{B}{s + \sqrt{\alpha\beta}}$ with

$$A = \left.\frac{sx_0 - \beta y_0}{s + \sqrt{\alpha\beta}}\right|_{s = \sqrt{\alpha\beta}} = \frac{\sqrt{\alpha\beta}\, x_0 - \beta y_0}{2\sqrt{\alpha\beta}} = \frac{1}{2}\left(x_0 - y_0\sqrt{\frac{\beta}{\alpha}}\right), \quad B = \left.\frac{sx_0 - \beta y_0}{s - \sqrt{\alpha\beta}}\right|_{s = -\sqrt{\alpha\beta}} = \frac{1}{2}\left(x_0 + y_0\sqrt{\frac{\beta}{\alpha}}\right).$$

It follows that $x(t) = Ae^{\sqrt{\alpha\beta}\, t} + Be^{-\sqrt{\alpha\beta}\, t}$. We obtain a similar formula for $y(t)$ (with x_0 and y_0 as well as α and β swapped for each other).

Solution. (without Laplace transforms) Our goal is to write down a single DE that only involves, say, $x(t)$. From the first DE, we get $y(t) = -\frac{1}{\beta}x'(t)$. Hence, $y'(t) = -\frac{1}{\beta}x''(t)$. Using that in the second DE, we obtain $-\frac{1}{\beta}x''(t) = -\alpha x(t)$ or, equivalently, $x''(t) - \alpha\beta x(t) = 0$.

Observe that, since $y(t) = -\frac{1}{\beta}x'(t)$, the initial condition $y(0) = y_0$ translates into $x'(0) = -\beta y_0$.

The roots are $\pm r$ where $r = \sqrt{\alpha\beta}$. Hence, $x(t) = C_1e^{rt} + C_2e^{-rt}$.

Using the initial conditions $x(0) = x_0$ and $x'(0) = -\beta y_0$, we find $C_1 + C_2 = x_0$ and $rC_1 - rC_2 = -\beta y_0$.

This results in $C_1 = \frac{1}{2}\left(x_0 - \frac{\beta y_0}{r}\right)$ and $C_2 = \frac{1}{2}\left(x_0 + \frac{\beta y_0}{r}\right)$. Correspondingly, using $r = \sqrt{\alpha\beta}$,

$$x(t) = \frac{1}{2}\left(x_0 - y_0\sqrt{\frac{\beta}{\alpha}}\right)e^{\sqrt{\alpha\beta}\, t} + \frac{1}{2}\left(x_0 + y_0\sqrt{\frac{\beta}{\alpha}}\right)e^{-\sqrt{\alpha\beta}\, t}$$

with a similar formula for $y(t) = -\frac{1}{\beta}x'(t)$.

Comment. The formulas take a particularly pleasing form when written in terms of cosh and sinh instead:

$$x(t) = x_0\cosh(\sqrt{\alpha\beta}\, t) - y_0\sqrt{\frac{\beta}{\alpha}}\sinh(\sqrt{\alpha\beta}\, t), \qquad y(t) = y_0\cosh(\sqrt{\alpha\beta}\, t) - x_0\sqrt{\frac{\alpha}{\beta}}\sinh(\sqrt{\alpha\beta}\, t).$$

Example 165. Determine conditions on x_0, y_0 (size of forces) and α, β (effectiveness of forces) that allow us to conclude who will win the battle.

Solution. Instead of analyzing our explicit formulas to find out which of $x(t)$ and $y(t)$ becomes 0 first (and therefore loses the battle), we make the following mathematical observation: the DEs dictate that, while fighting, both $x(t)$ and $y(t)$ are decreasing. On the other hand, purely mathematically, once one of the two turns negative then the DEs dictate that the other will increase while the negative one continues decreasing. Therefore, a force wins when its mathematical formula is increasing for large t.

Both solutions are combinations of $e^{\sqrt{\alpha\beta}t}$ and $e^{-\sqrt{\alpha\beta}t}$. Clearly, the term $e^{\sqrt{\alpha\beta}t}$ dominates the other as t gets large. For $x(t)$ that coefficient is $\frac{1}{2}(x_0 - y_0\sqrt{\beta/\alpha})$. This allows us to conclude that $x(t)$ wins the battle if $x_0 - y_0\sqrt{\frac{\beta}{\alpha}} > 0$. This is equivalent to $\alpha x_0^2 > \beta y_0^2$.

Solution. (without solving the DE) As an alternative, we can also start fresh and divide the two equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\beta y, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = -\alpha x$$

to get $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\alpha x}{\beta y}$. Using separation of variables, we find $\beta y\,\mathrm{d}y = \alpha x\,\mathrm{d}x$ which implies $\frac{1}{2}\beta y^2 = \frac{1}{2}\alpha x^2 + D$.

Consequently, $\alpha x^2 - \beta y^2 = C$ where $C = -2D$ is a constant. Using the initial conditions, we find $C = \alpha x_0^2 - \beta y_0^2$.

If $y(t_1) = 0$ (meaning that x wins at time t_1), then $\alpha x(t_1)^2 = C > 0$. On the other hand, if $x(t_1) = 0$, then $-\beta y(t_1)^2 = C < 0$. In other words, the sign of C determines who will win the battle.

Namely, x will win if $C > 0$ which is equivalent to $\alpha x_0^2 > \beta y_0^2$.

Conclusion. The condition we found is known as **Lanchester's square law**: its crucial message is that the sizes x_0, y_0 of the forces count quadratically, whereas the fighting effectivenesses α, β only count linearly. In other words, to beat a force with twice the effectiveness the other side only needs to have a force that is about 41.4% larger (since $\sqrt{2} \approx 1.4142$). Or, put differently, to beat a force of twice the size, the other side would need a fighting effectiveness that is more than 4 times as large.

Application to epidemiology: SIR model

The next example application results in a system of **nonlinear** differential equations. We do not have the tools to solve such equations.

Example 166. (epidemiology) Let us indicate the popular SIR model for short outbreaks of diseases among a population of constant size N.

In a SIR model, the population is compartmentalized into $S(t)$ susceptible, $I(t)$ infected and $R(t)$ recovered (or resistant) individuals ($N = S(t) + I(t) + R(t)$). In the Kermack-McKendrick model, the outbreak of a disease is modeled by

$$\frac{\mathrm{d}R}{\mathrm{d}t} = \gamma I, \quad \frac{\mathrm{d}S}{\mathrm{d}t} = -\beta S I, \quad \frac{\mathrm{d}I}{\mathrm{d}t} = \beta S I - \gamma I,$$

with γ modeling the recovery rate and β the infection rate. Note that this is a non-linear system of differential equations. For more details and many variations used in epidemiology, see:

https://en.wikipedia.org/wiki/Compartmental_models_in_epidemiology

Comment. The following variation

$$\frac{\mathrm{d}R}{\mathrm{d}t} = \gamma I R, \quad \frac{\mathrm{d}S}{\mathrm{d}t} = -\beta S I, \quad \frac{\mathrm{d}I}{\mathrm{d}t} = \beta S I - \gamma I R,$$

which assumes "infectious recovery", was used in 2014 to predict that facebook might lose 80% of its users by 2017. It is that claim, not mathematics (or even the modeling), which attracted a lot of media attention.

http://blogs.wsj.com/digits/2014/01/22/controversial-paper-predicts-facebook-decline/

The fin equation from thermodynamics

The following is an example from thermodynamics. The governing differential equation is a second-order DE that is like the equation describing the motion of a mass on a spring ($my'' + ky = 0$) except that one term has the opposite sign. Besides showcasing an application, we want to show off how $\cosh$ and $\sinh$ are useful for writing certain solutions in a more pleasing form.

Let $T(x)$ describe the temperature at position x in a fin with fin base at $x = 0$ and fin tip at $x = L$.

For more context on fins: `https://en.wikipedia.org/wiki/Fin_(extended_surface)`

If we write $\theta(x) = T(x) - T_\infty$ for the temperature excess at position x (with T_∞ the external temperature), then we find (under various simplifying assumptions) that the temperature distribution in our fin satisfies the following DE, known as the **fin equation**:

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}x^2} - m^2\theta = 0, \quad m^2 = \frac{hP}{kA} > 0.$$

- A is the cross-sectional area of the fin (assumed to be the same for all positions x).
- P is the perimeter of the fin (assumed to be the same for all positions x).
- k is the thermal conductivity of the material (assumed to be constant).
- h is the convection heat transfer coefficient (assumed to be constant).

Since the DE is homogeneous and linear with characteristic roots $\pm m$, the general solution is

$$\theta(x) = C_1 e^{mx} + C_2 e^{-mx} = D_1 \cosh(mx) + D_2 \sinh(mx).$$

The constants C_1, C_2 (or, equivalently, D_1, D_2) can then be found by imposing appropriate boundary conditions at the **fin base** ($x = 0$) and at the **fin tip** ($x = L$).

In practice, we often know the temperature at the fin base and therefore the temperature excess, resulting in the boundary condition $\theta(0) = \theta_0$. At the fin tip, common boundary conditions are:

- $\theta(L) \to 0$ as $L \to \infty$ (infinitely long fin)

  In this case, the fin is so long that the temperature at the fin tip approaches the external temperature. Mathematically, we get $\theta(x) = Ce^{-mx}$ since $e^{mx} \to \infty$ as $x \to \infty$. It follows from $\theta(0) = \theta_0$ that $C = \theta_0$.

  Thus, the temperature excess is $\theta(x) = \theta_0 e^{-mx}$.

- $\theta'(L) = 0$ (neglible heat loss at the fin tip, "adiabatic fin tip")

  This can be a more reasonable assumption than the infinitely long fin. Note that the total heat transfer from the fin is proportional to its surface area. If the surface area at the fin tip is a negligible fraction of the total surface area, then it is reasonable to assume that $\theta'(L) = 0$.

  In this case, the temperature excess is $\theta(x) = \theta_0 \frac{\cosh(m(L-x))}{\cosh(mL)}$.

  Check! Instead of computing this from scratch (do that as well, later!), check that this indeed solves the DE as well as the boundary conditions $\theta(0) = \theta_0$ and $\theta'(L) = 0$. This should be a rather quick check!

- $\theta(L) = \theta_L$ (specified temperature at fin tip)

  In this case, the temperature excess is $\theta(x) = \frac{\theta_L \sinh(mx) + \theta_0 \sinh(m(L-x))}{\sinh(mL)}$.

  Check! Again, check that this indeed solves the DE as well as the boundary conditions $\theta(0) = \theta_0$ and $\theta(L) = \theta_L$. Once more, this should be a quick and pleasant check.