

## Hyperbolic sine and cosine

**Review.** Euler's formula states that  $e^{it} = \cos(t) + i \sin(t)$ .

Recall that a function  $f(t)$  is **even** if  $f(-t) = f(t)$ . Likewise, it is **odd** if  $f(-t) = -f(t)$ .

Note that  $f(t) = t^n$  is even if and only if  $n$  is even. Likewise,  $f(t) = t^n$  is odd if and only if  $n$  is odd. That's where the names are coming from.

Any function  $f(t)$  can be decomposed into an even and an odd part as follows:

$$f(t) = f_{\text{even}}(t) + f_{\text{odd}}(t), \quad f_{\text{even}}(t) = \frac{1}{2}(f(t) + f(-t)), \quad f_{\text{odd}}(t) = \frac{1}{2}(f(t) - f(-t)).$$

Verify that  $f_{\text{even}}(t)$  indeed is even, and that  $f_{\text{odd}}(t)$  indeed is an odd function (regardless of  $f(t)$ ).

**Example 131.** The **hyperbolic cosine**, denoted  $\cosh(t)$ , is the even part of  $e^t$ . Likewise, the **hyperbolic sine**, denoted  $\sinh(t)$ , is the odd part of  $e^t$ .

- Equivalently,  $\cosh(t) = \frac{1}{2}(e^t + e^{-t})$  and  $\sinh(t) = \frac{1}{2}(e^t - e^{-t})$ .

- In particular,  $e^t = \cosh(t) + \sinh(t)$ .

As recalled above, any function is the sum of its even and odd part.

Comparing with Euler's formula, we find  $\cosh(it) = \cos(t)$  and  $\sinh(it) = i \sin(t)$ . This indicates that  $\cosh$  and  $\sinh$  are related to  $\cos$  and  $\sin$ , as their name suggests (see below for the "hyperbolic" part).

- $\frac{d}{dt} \cosh(t) = \sinh(t)$  and  $\frac{d}{dt} \sinh(t) = \cosh(t)$ .

- $\cosh(t)$  and  $\sinh(t)$  both satisfy the DE  $y'' = y$ .

We can write the general solution as  $C_1 e^t + C_2 e^{-t}$  or, if useful, as  $C_1 \cosh(t) + C_2 \sinh(t)$ .

- $\cosh(t)^2 - \sinh(t)^2 = 1$

Verify this by substituting  $\cosh(t) = \frac{1}{2}(e^t + e^{-t})$  and  $\sinh(t) = \frac{1}{2}(e^t - e^{-t})$ .

Note that the equation  $x^2 - y^2 = 1$  describes a **hyperbola** (just like  $x^2 + y^2 = 1$  describes a circle).

The above equation is saying that  $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cosh(t) \\ \sinh(t) \end{bmatrix}$  is a parametrization of the hyperbola.

**Comment.** Circles and hyperbolas are conic sections (as are ellipses and parabolas).

**Comment.** Hyperbolic geometry plays an important role, for instance, in special relativity:

[https://en.wikipedia.org/wiki/Hyperbolic\\_geometry](https://en.wikipedia.org/wiki/Hyperbolic_geometry)

**Homework.** Write down the parallel properties of cosine and sine.

