

We now revisit and continue Example 118:

**Example 127.** Consider two brine tanks. Initially, tank  $T_1$  is filled with 24gal water containing 3lb salt, and tank  $T_2$  with 9gal pure water.

- $T_1$  is being filled with 54gal/min water containing 0.5lb/gal salt.
- 72gal/min well-mixed solution flows out of  $T_1$  into  $T_2$ .
- 18gal/min well-mixed solution flows out of  $T_2$  into  $T_1$ .
- Finally, 54gal/min well-mixed solution is leaving  $T_2$ .

How much salt is in the tanks after  $t$  minutes?

**Solution.** Note that the amount of water in each tank is constant because the flows balance each other.

Let  $y_i(t)$  denote the amount of salt (in lb) in tank  $T_i$  after time  $t$  (in min). In the time interval  $[t, t + \Delta t]$ :

$$\Delta y_1 \approx 54 \cdot \frac{1}{2} \cdot \Delta t - 72 \cdot \frac{y_1}{24} \cdot \Delta t + 18 \cdot \frac{y_2}{9} \cdot \Delta t, \text{ so } y_1' = 27 - 3y_1 + 2y_2. \text{ Also, } y_1(0) = 3.$$

$$\Delta y_2 \approx 72 \cdot \frac{y_1}{24} \cdot \Delta t - (18 + 54) \cdot \frac{y_2}{9} \cdot \Delta t, \text{ so } y_2' = 3y_1 - 8y_2. \text{ Also, } y_2(0) = 0.$$

In conclusion, we have obtained the system of equations

$$\begin{aligned} y_1' &= -3y_1 + 2y_2 + 27, & y_1(0) &= 3, \\ y_2' &= 3y_1 - 8y_2, & y_2(0) &= 0. \end{aligned}$$

One strategy to solve this system is to first combine the two DEs to get a single equation for  $y_1$ .

- From the first DE, we get  $y_2 = \frac{1}{2}y_1' + \frac{3}{2}y_1 - \frac{27}{2}$ .
- Using this in the second DE, we obtain  $\left(\frac{1}{2}y_1' + \frac{3}{2}y_1 - \frac{27}{2}\right)' = 3y_1 - 8\left(\frac{1}{2}y_1' + \frac{3}{2}y_1 - \frac{27}{2}\right)$ .  
Simplified, this is  $y_1'' + 11y_1' + 18y_1 = 216$ .
- We already have the initial condition  $y_1(0) = 3$ . We get a second one by combining  $y_2 = \frac{1}{2}y_1' + \frac{3}{2}y_1 - \frac{27}{2}$  with  $y_2(0) = 0$  to get  $0 = y_2(0) = \frac{1}{2}y_1'(0) + \frac{3}{2}y_1(0) - \frac{27}{2} = \frac{1}{2}y_1'(0) - 9$ , which simplifies to  $y_1'(0) = 18$ .
- The IVP  $y_1'' + 11y_1' + 18y_1 = 216$  with initial conditions  $y_1(0) = 3$  and  $y_1'(0) = 18$  is one that we can solve!
  - The general solution of the corresponding homogeneous equation is  $y_h = C_1e^{-2t} + C_2e^{-9t}$ .
  - The simplest particular solution is of the form  $y_p = C$ . Plugging into the DE, we find  $y_p = \frac{216}{18} = 12$ .
  - Hence, the general solution to the (inhomogeneous) DE is  $y(x) = 12 + C_1e^{-2t} + C_2e^{-9t}$ .  
We then use the initial conditions  $y(0) = 12 + C_1 + C_2 \stackrel{!}{=} 3$ ,  $y'(0) = -2C_1 - 9C_2 \stackrel{!}{=} 18$  to find that for the unique solution of the IVP  $C_1 = -9$ ,  $C_2 = 0$ .

It has the unique solution  $y_1(t) = 12 - 9e^{-2t}$ .

- It follows that  $y_2 = \frac{1}{2}y_1' + \frac{3}{2}y_1 - \frac{27}{2} = \frac{9}{2} - \frac{9}{2}e^{-2t}$ .

**Note.** We could have found a particular solution with less calculations by observing (looking at the characteristic roots of the homogeneous DE and the inhomogeneous part) that there must be a solution of the form  $\mathbf{y}_p(t) = \mathbf{a}$ . We can then find  $\mathbf{a}$  by plugging into the differential equation. However, noticing that, for a constant solution, each tank has to have a constant concentration of 0.5lb/gal of salt, we find  $\mathbf{y}_p(t) = \begin{bmatrix} 12 \\ 4.5 \end{bmatrix}$ .

**In matrix-vector form.** If we write  $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ , our system takes the form

$$\mathbf{y}' = \begin{bmatrix} -3 & 2 \\ 3 & -8 \end{bmatrix} \mathbf{y} + \begin{bmatrix} 27 \\ 0 \end{bmatrix}, \quad \mathbf{y}(0) = \begin{bmatrix} 3 \\ 0 \end{bmatrix}.$$

**Example 128. (extra)** Three brine tanks  $T_1, T_2, T_3$ .

$T_1$  contains 20gal water with 10lb salt,  $T_2$  40gal pure water,  $T_3$  50gal water with 30lb salt.

$T_1$  is filled with 10gal/min water with 2lb/gal salt. 10gal/min well-mixed solution flows out of  $T_1$  into  $T_2$ . Also, 10gal/min well-mixed solution flows out of  $T_2$  into  $T_3$ . Finally, 10gal/min well-mixed solution is leaving  $T_3$ . How much salt is in the tanks after  $t$  minutes?

**Solution.** Let  $y_i(t)$  denote the amount of salt (in lb) in tank  $T_i$  after time  $t$  (in min).

In the time interval  $[t, t + \Delta t]$ :

$$\Delta y_1 \approx 10 \cdot 2 \cdot \Delta t - 10 \frac{y_1}{20} \cdot \Delta t, \text{ so } y_1' = 20 - \frac{1}{2}y_1. \text{ Also, } y_1(0) = 10.$$

$$\Delta y_2 \approx 10 \cdot \frac{y_1}{20} \cdot \Delta t - 10 \frac{y_2}{40} \cdot \Delta t, \text{ so } y_2' = \frac{1}{2}y_1 - \frac{1}{4}y_2. \text{ Also, } y_2(0) = 0.$$

$$\Delta y_3 \approx 10 \cdot \frac{y_2}{40} \cdot \Delta t - 10 \frac{y_3}{50} \cdot \Delta t, \text{ so } y_3' = \frac{1}{4}y_2 - \frac{1}{5}y_3. \text{ Also, } y_3(0) = 30.$$

Using matrix notation and writing  $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ , this is  $\mathbf{y}' = \begin{bmatrix} -1/2 & 0 & 0 \\ 1/2 & -1/4 & 0 \\ 0 & 1/4 & -1/5 \end{bmatrix} \mathbf{y} + \begin{bmatrix} 20 \\ 0 \\ 0 \end{bmatrix}$ ,  $\mathbf{y}(0) = \begin{bmatrix} 10 \\ 0 \\ 30 \end{bmatrix}$ .

We can actually solve this IVP!

[Do it! Start by finding  $y_1$  from the first DE, then move on to  $y_2 \dots$ ]

Here, we content ourselves with finding a particular solution (and ignoring the initial conditions). The method of undetermined coefficients tells us that there is a solution of the form  $\mathbf{y}_p(t) = \mathbf{a}$ . Of course, we can find  $\mathbf{a}$  by plugging into the differential equation. However, noticing that, for a constant solution, each tank has to have a concentration of 2lb/gal of salt, we find  $\mathbf{y}_p = (40, 80, 100)$  without calculation.