Example 119. Consider two brine tanks. Initially, tank T_1 is filled with 10gal water containing 2lb salt, and tank T_2 with 5gal pure water.

- T_1 is being filled with 4gal/min water containing 0.5lb/gal salt.
- 5gal/min well-mixed solution flows out of T_1 into T_2.
- 2gal/min well-mixed solution flows out of T_2 into T_1.
- Finally, 1gal/min well-mixed solution is leaving T_2.

Derive a system of equations for the amount of salt in the tanks after t minutes.

Solution. Let $V_i(t)$ denote the amount of solution (in gal) in tank T_i after time t (in min). Then $V_1(t) = 10 + 4t - 5t + 2t = 10 + t$ while $V_2(t) = 5 + 5t - 2t - t = 5 + 2t$.

Let $y_i(t)$ denote the amount of salt (in lb) in tank T_i after time t (in min). In the time interval $[t, t + \Delta t]$:

$\Delta y_1 \approx 4 \cdot \frac{1}{2} \cdot \Delta t - 5 \cdot \frac{y_1}{V_1} \cdot \Delta t + 2 \cdot \frac{y_2}{V_2} \cdot \Delta t$, so $y_1' = 2 - 5\frac{y_1}{V_1} + 2\frac{y_2}{V_2}$. Also, $y_1(0) = 2$.

$\Delta y_2 \approx 5 \cdot \frac{y_1}{V_1} \cdot \Delta t - (2 + 1) \cdot \frac{y_2}{V_2} \cdot \Delta t$, so $y_2' = 5\frac{y_1}{V_1} - 3\frac{y_2}{V_2}$. Also, $y_2(0) = 0$.

In conclusion, we have obtained the system of equations

$$\begin{aligned} y_1' &= -\frac{5}{10+t} y_1 + \frac{2}{5+2t} y_2 + 2, \qquad & y_1(0) = 2, \\ y_2' &= \frac{5}{10+t} y_1 - \frac{3}{5+2t} y_2, & y_2(0) = 0. \end{aligned}$$

Note that this is a system of linear DEs. It is inhomogeneous (because of the $+2$ in the first equation). Its coefficients are not constant.

In matrix-vector form. If we write $\boldsymbol{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, then the system becomes

$$\boldsymbol{y}' = \begin{bmatrix} -\frac{5}{10+t} & \frac{2}{5+2t} \\ \frac{5}{10+t} & -\frac{3}{5+2t} \end{bmatrix} \boldsymbol{y} + \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \quad \boldsymbol{y}(0) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}.$$

Higher-order linear DEs as first-order systems

The following examples show that any higher-order DE can be converted to a system of first-order DEs. This illustrates why we care about systems of DEs, even if we work with only one function. It is also the reason why we looked at results like the uniqueness and existence theorem or Euler's method only for first-order DEs.

These results can be naturally generalized from a single DE to a system of DEs.

Example 120. Write the (second-order) differential equation $y'' = 2y' + 5y$ as a system of (first-order) differential equations.

Solution. Write $y_1 = y$ and $y_2 = y'$. Then $y'' = 2y' + 5y$ becomes $y_2' = 2y_2 + 5y_1$.

Therefore, $y'' = 2y' + 5y$ translates into the first-order system $\begin{cases} y_1' = y_2 \\ y_2' = 5y_1 + 2y_2 \end{cases}$.

In matrix-vector form, this is $\boldsymbol{y}' = \begin{bmatrix} 0 & 1 \\ 5 & 2 \end{bmatrix} \boldsymbol{y}$.

Advanced comment. Here, we only use the matrix-vector notation as a device for writing the system of equations in a more compact form. However, it turns out that the matrix-vector notation makes certain techniques more transparent (just like writing a system of equations in the form $A\boldsymbol{x} = \boldsymbol{b}$ suggests introducing the matrix inverse to simply write $\boldsymbol{x} = A^{-1}\boldsymbol{b}$). For instance, the unique solution to a homogeneous linear system $\boldsymbol{y}' = M\boldsymbol{y}$ (where M is a matrix with constant entries) with initial condition $\boldsymbol{y}(0) = \boldsymbol{c}$ can be expressed as $\boldsymbol{y}(x) = e^{Mx}\boldsymbol{c}$, just as in the case of a single linear DE. Here, e^{Mx} is the **matrix exponential**. This will be one of the topics discussed in both Differential Equations II and Linear Algebra II.

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Example 121. Write the (third-order) differential equation $y''' = 3y'' - 2y' + 4y$ as a system of (first-order) differential equations.

Solution. Write $y_1 = y$, $y_2 = y'$ and $y_3 = y''$.

Then, $y''' = 3y'' - 2y' + 4y$ translates into the first-order system $\begin{cases} y_1' = y_2 \\ y_2' = y_3 \\ y_3' = 4y_1 - 2y_2 + 3y_3 \end{cases}$.

In matrix-vector form, this is $\boldsymbol{y}' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -2 & 3 \end{bmatrix} \boldsymbol{y}$.

Example 122. Consider the following system of (second-order) initial value problems:

$$\begin{array}{l} y_1'' = 2y_1' - 3y_2' + 7y_2 \\ y_2'' = 4y_1' + y_2' - 5y_1 \end{array} \qquad y_1(0) = 2,\ y_1'(0) = 3,\ y_2(0) = -1,\ y_2'(0) = 1$$

Write it as a first-order initial value problem in the form $\boldsymbol{y}' = M\boldsymbol{y}$, $\boldsymbol{y}(0) = \boldsymbol{y}_0$.

Solution. Introduce $y_3 = y_1'$ and $y_4 = y_2'$. Then, the given system translates into

$$\boldsymbol{y}' = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 7 & 2 & -3 \\ -5 & 0 & 4 & 1 \end{bmatrix} \boldsymbol{y}, \quad \boldsymbol{y}(0) = \begin{bmatrix} 2 \\ -1 \\ 3 \\ 1 \end{bmatrix}.$$