Example 109. The motion of a mass on a spring under an external force is described by $my'' + 4y = 3\cos(t) - \cos(2t) + 7\cos(3t)$. For which values of $m > 0$ does resonance occur?

Solution. The characteristic roots of the homogeneous DE are $\pm i\sqrt{\frac{4}{m}}$ so that the natural frequency is $\sqrt{\frac{4}{m}}$. The external frequencies are $\lambda = 1, 2, 3$. Hence, resonance occurs when $\sqrt{\frac{4}{m}} = \lambda$ for $\lambda \in \{1, 2, 3\}$. This is equivalent to $m = \frac{4}{\lambda^2}$ so that this happens if $m = 4$, $m = 1$, $m = \frac{4}{9}$.

Comment. The external force $3\cos(t) - \cos(2t) + 7\cos(3t)$ may look artifical. However, that is not the case! Indeed, essentially any 2π-periodic function can be written as an (infinite) combination of such cosine and sine terms. The resulting series are known as **Fourier series**.

External forces plus damping

In the presence of both damping ($d > 0$) and a periodic external force ($f(t)$), the motion $y(t)$ of a mass on a spring is described by the DE

$$my'' + dy' + ky = f(t).$$

Solving the DE, we find that $y(t)$ splits into **transient motion** y_{tr} (with $y_{\text{tr}}(t) \to 0$ as $t \to \infty$) and **steady periodic oscillations** y_{sp}:

$$y(t) = y_{\text{tr}} + y_{\text{sp}}.$$

The following example spells this out.

Comment. Note that y_{sp} will correspond to the simplest particular solution, while y_{tr} corresponds to the solution of the corresponding homogeneous system (where we have no external force).

Example 110. A forced mechanical oscillator is described by $2y'' + 2y' + y = 10\sin(t)$. As $t \to \infty$, what are the period and the amplitude of the resulting steady periodic oscillations?

Solution. The characteristic roots of the homogeneous DE are $\frac{1}{4}(-2 \pm \sqrt{4-8}) = -\frac{1}{2} \pm \frac{1}{2}i$. Accordingly, the system without external force is underdamped (because of the $\pm i/2$ the solutions will involve oscillations).

The characteristic roots for the inhomogeneous part are $\pm i$ so that there must be a particular solution $y_p = A\cos(t) + B\sin(t)$ with coefficients A, B that we need to determine by plugging into the DE. This results in $A = -4$ and $B = -2$ (do it!).

Hence, the general solution is $y(t) = \underbrace{-4\cos(t) - 2\sin(t)}_{y_{\text{sp}}} + \underbrace{e^{-t/2}\left(C_1\cos\left(\frac{t}{2}\right) + C_2\sin\left(\frac{t}{2}\right)\right)}_{y_{\text{tr}} \to 0 \text{ as } t \to \infty}$.

The period of $y_{\text{sp}} = -4\cos(t) - 2\sin(t)$ is 2π and the amplitude is $\sqrt{(-4)^2 + (-2)^2} = \sqrt{20}$.

Comment. Using the polar coordinates $(-4, -2) = \sqrt{20}(\cos\alpha, \sin\alpha)$ where $\alpha = \tan^{-1}(1/2) + \pi \approx 3.605$, we can alternatively express the steady periodic oscillations as $y_{\text{sp}} = -4\cos(t) - 2\sin(t) = \sqrt{20}(\cos(t - \alpha))$.

Example 111. A forced mechanical oscillator is described by $y'' + 5y' + 6y = 2\cos(3t)$. What are the (circular) frequency and the amplitude of the resulting steady periodic oscillations?

Solution. The characteristic roots of the homogeneous DE are $-2, -3$. Accordingly, the system without external force is overdamped (the solutions will not involve oscillations).

The characteristic roots for the inhomogeneous part are $\pm 3i$ so that there must be a particular solution $y_p = A\cos(3t) + B\sin(3t)$ with coefficients A, B that we need to determine by plugging into the DE. To do so, we compute $y_p' = -3A\sin(3t) + 3B\cos(3t)$ as well as $y_p'' = -9A\cos(3t) - 9B\sin(3t)$.

$$\begin{aligned} y_p'' + 5y_p' + 6y_p &= (-9A\cos(3t) - 9B\sin(3t)) + 5(-3A\sin(3t) + 3B\cos(3t)) + 6(A\cos(3t) + B\sin(3t)) \\ &= (-9A + 15B + 6A)\cos(3t) + (-9B - 15A + 6B)\sin(3t) \\ &\overset{!}{=} 2\cos(3t) \end{aligned}$$

This results in the two equations $-3A + 15B = 2$ and $-3B - 15A = 0$, which we solve to find $A = -\frac{1}{39}$ and $B = \frac{5}{39}$.

The general solution is $y(t) = \underbrace{-\frac{1}{39}\cos(3t) + \frac{5}{39}\sin(3t)}_{y_{\text{sp}}} + \underbrace{C_1 e^{-2t} + C_2 e^{-3t}}_{y_{\text{tr}} \to 0 \text{ as } t \to \infty}$.

The frequency of $y_{\text{sp}} = -\frac{1}{39}\cos(3t) + \frac{5}{39}\sin(3t)$ is 3 and the amplitude is $\sqrt{\left(-\frac{1}{39}\right)^2 + \left(\frac{5}{39}\right)^2} = \sqrt{\frac{2}{117}}$.

Example 112. Find the steady periodic solution to $y'' + 2y' + 5y = \cos(\lambda t)$. What is the amplitude of the steady periodic oscillations? For which λ is the amplitude maximal?

Solution. The characteristic roots of the homogeneous DE are $-1 \pm 2i$.

[Not really needed, because positive damping prevents duplication; can you see it?]

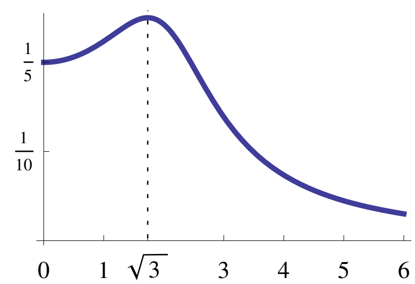
Hence, $y_{\text{sp}} = A\cos(\lambda t) + B\sin(\lambda t)$ and to find A, B we need to plug into the DE.

Doing so, we find $A = \frac{5 - \lambda^2}{(5 - \lambda^2)^2 + 4\lambda^2}$, $B = \frac{2\lambda}{(5 - \lambda^2)^2 + 4\lambda^2}$.

Thus, the amplitude of y_{sp} is $r(\lambda) = \sqrt{A^2 + B^2} = \frac{1}{\sqrt{(5 - \lambda^2)^2 + 4\lambda^2}}$.

The function $r(\lambda)$ is sketched to the right. It has a maximum at $\lambda = \sqrt{3}$ at which the amplitude is unusually large (well, here it is not very pronounced). We say that **practical resonance** occurs for $\lambda = \sqrt{3}$.

[For comparison, without damping, (pure) resonance occurs for $\lambda = \sqrt{5}$.]



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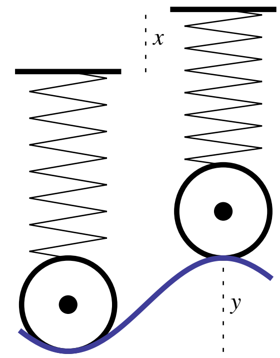
Example 113. (extra) A car is going at constant speed v on a washboard road surface ("bumpy road") with height profile $y(s) = a\cos\left(\frac{2\pi s}{L}\right)$. Assume that the car oscillates vertically as if on a spring (no dashpot). Describe the resulting oscillations.



The first "car" is assumed to be in equilibrium.

Solution. With x as in the sketch, the spring is stretched by $x - y$. Hence, by Hooke's and Newton's laws, its motion is described by $mx'' = -k(x-y)$.

At constant speed, $s = vt$ and we obtain the DE $mx'' + kx = ky = ka\cos\left(\frac{2\pi vt}{L}\right)$, which is inhomogeneous linear with constant coefficients. Let's solve it.

The natural frequency is $\omega_0 = \sqrt{\frac{k}{m}}$.

The external frequency is $i\frac{2\pi v}{L} = \pm i\omega$. $\omega = \frac{2\pi v}{L}$.

Case 1: $\omega \neq \omega_0$. Then a particular solution is $x_p = b_1\cos(\omega t) + b_2\sin(\omega t) = A\cos(\omega t - \alpha)$ for unique values of b_1, b_2 (which we do not compute here). The general solution is of the form $x = x_p + C_1\cos(\omega_0 t) + C_2\sin(\omega_0 t)$.

Case 2: $\omega = \omega_0$. Then a particular solution is $x_p = t[b_1\cos(\omega t) + b_2\sin(\omega t)] = At\cos(\omega t - \alpha)$ for unique values of b_1, b_2 (which we do not compute). Note that the amplitude in x_p increases without bound; the same is true for the general solution $x = x_p + C_1\cos(\omega_0 t) + C_2\sin(\omega_0 t)$. This phenomenon is resonance; it occurs if an external frequency matches a natural frequency.