Adding external forces and the phenomenon of resonance

The motion of a mass m on a spring, with damping and with an external force $f(t)$ taken into account, can be modeled by the DE

$$my'' + dy' + ky = f(t).$$

Note that each term is representing a force: $my'' = ma$ is the force due to Newton's second law ($F = ma$), the term dy' models damping (proportional to the velocity), the term ky represents the force due to Hooke's law, and the term $f(t)$ represents an external force that acts on the mass at time t.

Example 105. Describe the solutions of $y'' + 4y = \cos(\lambda t)$. (Here, $\lambda > 0$ is a constant.)

Solution. The characteristic roots of the homogeneous DE are $\pm 2i$ so that 2 is the **natural frequency** (the frequency at which the system would oscillate in the absence of external forces; mathematically, this reflects the fact that the general solution to the corresponding homogeneous DE is $A\cos(2t) + B\sin(2t)$, which has frequency $\omega = 2$).

The characteristic roots of the inhomogeneous part are $\pm\lambda i$ where λ is the **external frequency**.

Case 1: $\lambda \neq 2$. Then there is a particular solution of the form $y_p = A\cos(\lambda t) + B\sin(\lambda t)$. To determine the unique values of A, B, we plug into the DE:

$$y_p'' + 4y_p = (4 - \lambda^2)A\cos(\lambda t) + (4 - \lambda^2)B\sin(\lambda t) \overset{!}{=} \cos(\lambda t)$$

We conclude that $(4 - \lambda^2)A = 1$ and $(4 - \lambda^2)B = 0$. Solving these, we find $A = 1/(4 - \lambda^2)$ and $B = 0$.

Thus, the general solution is of the form $y = \frac{1}{4 - \lambda^2}\cos(\lambda t) + C_1\cos(2t) + C_2\sin(2t)$.

Case 2: $\lambda = 2$. Now, there is a particular solution of the form $y_p = At\cos(2t) + Bt\sin(2t)$. To determine the unique values of A, B, we again plug into the DE (which is more work this time):

$$y_p'' + 4y_p \overset{\text{work}}{=} 4B\cos(2t) - 4A\sin(2t) \overset{!}{=} \cos(2t)$$

We conclude that $4B = 1$ and $-4A = 0$. Solving these, we find $A = 0$ and $B = 1/4$.

Thus, the general solution is of the form $y = \frac{1}{4}t\sin(2t) + C_1\cos(2t) + C_2\sin(2t)$.

Note that the amplitude in y_p increases without bound (so that the same is true for the general solution).

This phenomenon is called **resonance**; it occurs if an external frequency matches a natural frequency.

If an external frequency matches a natural frequency, then **resonance** occurs.

In that case, we obtain amplitudes that grow without bound.

Resonance (or anything close to it) is very important for practical purposes because large amplitudes can be very destructive: singing to shatter glass, earth quake waves and buildings, marching soldiers on bridges, ...

Comment. Mathematically speaking, resonance occurs if the characteristic roots of the homogeneous DE and the inhomogeneous part overlap. In that case, the solutions acquire a factor of the variable t (or x) which changes the nature of the solutions (and results in unbounded amplitudes).

Example 106. Consider $y'' + 9y = 10\cos(2\lambda t)$. For what value of λ does resonance occur?

Solution. The natural frequency is 3. The external frequency is 2λ. Hence, resonance occurs when $\lambda = \frac{3}{2}$.

Example 107. The motion of a mass on a spring under an external force is described by $5y'' + 2y = -2\sin(3\lambda t)$. For which value of λ does resonance occur?

Solution. The natural frequency is $\sqrt{\frac{2}{5}}$. The external frequency is 3λ. Hence, resonance occurs when $\lambda = \frac{1}{3}\sqrt{\frac{2}{5}}$.

Example 108. The motion of a mass on a spring under an external force is described by $3y'' + ky = \cos(t/2)$. For which value of $k > 0$ does resonance occur?

Solution. The natural frequency is $\sqrt{\frac{k}{3}}$. The external frequency is $\frac{1}{2}$. Hence, resonance occurs when $\sqrt{\frac{k}{3}} = \frac{1}{2}$. This happens if $k = 3 \cdot \left(\frac{1}{2}\right)^2 = \frac{3}{4}$.