**Example 91.** Consider the DE $y'' + 4y' + 4y = 2e^{3x} - 5e^{-2x}$.

(a) What is the simplest form (with undetermined coefficients) of a particular solution?

(b) Determine a particular solution using our results from Examples 89 and 90.

(c) Determine the general solution.

**Solution.**

(a) Note that $D^2 + 4D + 4 = (D+2)^2$.

| | homogeneous DE | inhomogeneous part |
|---|---|---|
| characteristic roots | $-2, -2$ | $3, -2$ |
| solutions | $e^{-2x}, xe^{-2x}$ | $e^{3x}, x^2e^{-2x}$ |

Hence, there has to be a particular solution of the form $y_p = Ae^{3x} + Bx^2e^{-2x}$.

To find the (unique) values of $A$ and $B$, we can plug into the DE. Alternatively, we can break the problem into two pieces as illustrated in the next part.

(b) Write the DE as $Ly = 2e^{3x} - 5e^{-2x}$ where $L = D^2 + 4D + 4$. In Example 89 we found that $y_1 = \frac{1}{25}e^{3x}$ satisfies $Ly_1 = e^{3x}$. Also, in Example 90 we found that $y_2 = \frac{7}{2}x^2e^{-2x}$ satisfies $Ly_2 = 7e^{-2x}$.

By linearity, it follows that $L(Ay_1 + By_2) = ALy_1 + BLy_2 = Ae^{3x} + 7Be^{-2x}$.

To get a particular solution $y_p$ of our DE, we need $A = 2$ and $7B = -5$.

Hence, $y_p = 2y_1 - \frac{5}{7}y_2 = \frac{2}{25}e^{3x} - \frac{5}{2}x^2e^{-2x}$.

**Comment.** Of course, if we hadn't previously solved Examples 89 and 90, we could have plugged the result from the first part into the DE to determine the coefficients $A$ and $B$. On the other hand, breaking the inhomogeneous part ($2e^{3x} - 5e^{-2x}$) up into pieces (here, $e^{3x}$ and $e^{-2x}$) can help keep things organized, especially when working by hand.

(c) The general solution is $\frac{2}{25}e^{3x} - \frac{5}{2}x^2e^{-2x} + (C_1 + C_2x)e^{2x}$.

**Example 92.** Consider the DE $y'' - 2y' + y = 5\sin(3x)$.

(a) What is the simplest form (with undetermined coefficients) of a particular solution?

(b) Determine a particular solution.

(c) Determine the general solution.

**Solution.** Note that $D^2 - 2D + 1 = (D-1)^2$.

| | homogeneous DE | inhomogeneous part |
|---|---|---|
| characteristic roots | $1, 1$ | $\pm 3i$ |
| solutions | $e^x, xe^x$ | $\cos(3x), \sin(3x)$ |

(a) This tells us that there exists a particular solution of the form $y_p = A\cos(3x) + B\sin(3x)$.

(b) To find the values of $A$ and $B$, we plug into the DE.

$y_p' = -3A\sin(3x) + 3B\cos(3x)$

$y_p'' = -9A\cos(3x) - 9B\sin(3x)$

$y_p'' - 2y_p' + y_p = (-8A - 6B)\cos(3x) + (6A - 8B)\sin(3x) \stackrel{!}{=} 5\sin(3x)$

Equating the coefficients of $\cos(x)$, $\sin(x)$, we obtain the two equations $-8A - 6B = 0$ and $6A - 8B = 5$.

Solving these, we find $A = \frac{3}{10}$, $B = -\frac{2}{5}$. Accordingly, a particular solution is $y_p = \frac{3}{10}\cos(3x) - \frac{2}{5}\sin(3x)$.

(c) The general solution is $y(x) = \frac{3}{10}\cos(3x) - \frac{2}{5}\sin(3x) + (C_1 + C_2x)e^x$.

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**Example 93.** Consider the DE $y'' - 2y' + y = 5e^{2x}\sin(3x) + 7xe^x$. What is the simplest form (with undetermined coefficients) of a particular solution?

**Solution.** Since $D^2 - 2D + 1 = (D-1)^2$, the characteristic roots are $1, 1$. The roots for the inhomogeneous part are $2 \pm 3i, 1, 1$. Hence, there has to be a particular solution of the form $y_p = Ae^{2x}\cos(3x) + Be^{2x}\sin(3x) + Cx^2e^x + Dx^3e^x$.

(We can then plug into the DE to determine the (unique) values of the coefficients $A, B, C, D$.)

**Example 94. (homework)** What is the shape of a particular solution of $y'' + 4y' + 4y = x\cos(x)$?

**Solution.** The characteristic roots are $-2, -2$. The roots for the inhomogeneous part are $\pm i, \pm i$. Hence, there has to be a particular solution of the form $y_p = (C_1 + C_2x)\cos(x) + (C_3 + C_4x)\sin(x)$.

**Continuing to find a particular solution.** To find the value of the $C_j$'s, we plug into the DE.

$y_p' = (C_2 + C_3 + C_4x)\cos(x) + (C_4 - C_1 - C_2x)\sin(x)$

$y_p'' = (2C_4 - C_1 - C_2x)\cos(x) + (-2C_2 - C_3 - C_4x)\sin(x)$

$y_p'' + 4y_p' + 4y_p = (3C_1 + 4C_2 + 4C_3 + 2C_4 + (3C_2 + 4C_4)x)\cos(x)$

$\quad + (-4C_1 - 2C_2 + 3C_3 + 4C_4 + (-4C_2 + 3C_4)x)\sin(x) \overset{!}{=} x\cos(x)$.

Equating the coefficients of $\cos(x)$, $x\cos(x)$, $\sin(x)$, $x\sin(x)$, we get the equations $3C_1 + 4C_2 + 4C_3 + 2C_4 = 0$, $3C_2 + 4C_4 = 1$, $-4C_1 - 2C_2 + 3C_3 + 4C_4 = 0$, $-4C_2 + 3C_4 = 0$.

Solving (this is tedious!), we find $C_1 = -\frac{4}{125}$, $C_2 = \frac{3}{25}$, $C_3 = -\frac{22}{125}$, $C_4 = \frac{4}{25}$.

Hence, $y_p = \left(-\frac{4}{125} + \frac{3}{25}x\right)\cos(x) + \left(-\frac{22}{125} + \frac{4}{25}x\right)\sin(x)$.

**Example 95. (homework)** What is the shape of a particular solution of $y'' + 4y' + 4y = 4e^{3x}\sin(2x) - x\sin(x)$.

**Solution.** The characteristic roots are $-2, -2$. The roots for the inhomogeneous part roots are $3 \pm 2i, \pm i, \pm i$. Hence, there has to be a particular solution of the form

$y_p = C_1e^{3x}\cos(2x) + C_2e^{3x}\sin(2x) + (C_3 + C_4x)\cos(x) + (C_5 + C_6x)\sin(x)$.

**Continuing to find a particular solution.** To find the values of $C_1, \ldots, C_6$, we plug into the DE. But this final step is so boring that we don't go through it here. Computers (currently?) cannot afford to be as selective; mine obediently calculated: $y_p = -\frac{4}{841}e^{3x}(20\cos(2x) - 21\sin(2x)) + \frac{1}{125}((-22 + 20x)\cos(x) + (4 - 15x)\sin(x))$

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