Example 86. (review) Find the general solution of $y^{(7)} + 8y^{(6)} + 42y^{(5)} + 104y^{(4)} + 169y''' = 0$.

Use the fact that $-2+3i$ is a repeated characteristic root.

Solution. The characteristic polynomial $p(D) = D^3(D^2+4D+13)^2$ has roots $0, 0, 0, -2 \pm 3i, -2 \pm 3i$.

[Since $-2+3i$ is a root so must be $-2-3i$. Repeating them once, together with $0, 0, 0$ results in 7 roots.]

Hence, the general solution is $(A + Bx + Cx^2) + (D + Ex)\, e^{-2x}\cos(3x) + (F + Gx)e^{-2x}\sin(3x)$.

Example 87. (review) Consider the function $y(x) = 7x - 5x^2 e^{4x}$. Find an operator $p(D)$ such that $p(D)y = 0$.

Comment. This is the same as determining a homogeneous linear DE with constant coefficients solved by $y(x)$.

Solution. In order for $y(x)$ to be a solution of $p(D)y = 0$, the characteristic roots must include $0, 0, 4, 4, 4$.

The simplest choice for $p(D)$ thus is $p(D) = D^2(D-4)^3$.

Inhomogeneous linear DEs: The method of undetermined coefficients

The **method of undetermined coefficients** allows us to solve certain inhomogeneous linear DEs $Ly = f(x)$ with constant coefficients..

It works if $f(x)$ is itself a solution of a homogeneous linear DE with constant coefficients (see previous example).

Example 88. Determine the general solution of $y'' + 4y = 12x$.

Solution. The DE is $p(D)y = 12x$ with $p(D) = D^2 + 4$, which has roots $\pm 2i$. Thus, the general solution is $y(x) = y_p(x) + C_1\cos(2x) + C_2\sin(2x)$. It remains to find a particular solution y_p.

Since $D^2 \cdot (12x) = 0$, we apply D^2 to both sides of the DE to get the **homogeneous** DE $D^2(D^2+4) \cdot y = 0$.

Its general solution is $C_1 + C_2x + C_3\cos(2x) + C_4\sin(2x)$ and y_p must be of this form. Indeed, there must be a particular solution of the simpler form $y_p = C_1 + C_2x$ (because $C_3\cos(2x) + C_4\sin(2x)$ can be added to any y_p).

It remains to find appropriate values C_1, C_2 such that $y_p'' + 4y_p = 12x$. Since $y_p'' + 4y_p = 4C_1 + 4C_2x$, comparing coefficients yields $4C_1 = 0$ and $4C_2 = 12$, so that $C_1 = 0$ and $C_2 = 3$. In other words, $y_p = 3x$.

Therefore, the general solution to the original DE is $y(x) = 3x + C_1\cos(2x) + C_2\sin(2x)$.

Example 89. Determine the general solution of $y'' + 4y' + 4y = e^{3x}$.

Solution. The DE is $p(D)y = e^{3x}$ with $p(D) = D^2 + 4D + 4 = (D+2)^2$, which has roots $-2, -2$. Thus, the general solution is $y(x) = y_p(x) + (C_1 + C_2x)e^{-2x}$. It remains to find a particular solution y_p.

Since $(D-3)e^{3x} = 0$, we apply $(D-3)$ to the DE to get the **homogeneous** DE $(D-3)(D+2)^2 y = 0$.

Its general solution is $(C_1 + C_2x)e^{-2x} + C_3e^{3x}$ and y_p must be of this form. Indeed, there must be a particular solution of the simpler form $y_p = Ae^{3x}$.

To determine the value of C, we plug into the original DE: $y_p'' + 4y_p' + 4y_p = (9 + 4 \cdot 3 + 4)Ae^{3x} \overset{!}{=} e^{3x}$. Hence, $A = 1/25$. Therefore, the general solution to the original DE is $y(x) = (C_1 + C_2x)e^{-2x} + \frac{1}{25}e^{3x}$.

Solution. (same, just shortened) In schematic form:

	homogeneous DE	inhomogeneous part
characteristic roots	$-2, -2$	3
solutions	e^{-2x}, xe^{-2x}	e^{3x}

This tells us that there exists a particular solution of the form $y_p = Ae^{3x}$. Then the general solution is

$$y = y_p + C_1e^{-2x} + C_2xe^{-2x}.$$

So far, we didn't need to do any calculations (besides determining the roots)! However, we still need to determine the value of A (by plugging into the DE as above), namely $A = \frac{1}{25}$. For this reason, this approach is often called the **method of undetermined coefficients**.

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We found the following recipe for solving nonhomogeneous linear DEs with constant coefficients:

That approach works for $p(D)y = f(x)$ whenever the right-hand side $f(x)$ is the solution of some homogeneous linear DE with constant coefficients: $q(D)f(x) = 0$

(method of undetermined coefficients) To find a particular solution y_p to an inhomogeneous linear DE with constant coefficients $p(D)y = f(x)$:

- Determine the characteristic roots of the homogeneous DE and corresponding solutions.
- Find the roots of $q(D)$ so that $q(D)f(x) = 0$. [This does not work for all $f(x)$.]

  Let $y_{p,1}, y_{p,2}, \ldots$ be the additional solutions (when the roots are added to those of the homogeneous DE).

Then there exists (unique) C_i so that

$$y_p = C_1 y_{p,1} + C_2 y_{p,2} + \ldots$$

To find the values C_i, we need to plug y_p into the original DE.

Why? To see that this approach works, note that applying $q(D)$ to both sides of the inhomogeneous DE $p(D)y = f(x)$ results in $q(D)p(D)y = 0$ which is homogeneous. We already know that the solutions to the homogeneous DE can be added to any particular solution y_p. Therefore, we can focus only on the additional solutions coming from the roots of $q(D)$.

For which $f(x)$ does this work? By Theorem 72, we know exactly which $f(x)$ are solutions to homogeneous linear DEs with constant coefficients: these are linear combinations of exponentials $x^j e^{rx}$ (which includes $x^j e^{ax}\cos(bx)$ and $x^j e^{ax}\sin(bx)$).

Example 90. Determine the general solution of $y'' + 4y' + 4y = 7e^{-2x}$.

Solution. The homogeneous DE is $y'' + 4y' + 4y = 0$ and the inhomogeneous part is $7e^{-2x}$.

	homogeneous DE	inhomogeneous part
characteristic roots	$-2, -2$	-2
solutions	e^{-2x}, xe^{-2x}	$x^2 e^{-2x}$

This tells us that there exists a particular solution of the form $y_p = Cx^2 e^{-2x}$. To find the value of C, we plug into the DE.

$y_p' = C(-2x^2 + 2x)e^{-2x}$

$y_p'' = C(4x^2 - 8x + 2)e^{-2x}$

$y_p'' + 4y_p' + 4y_p = 2Ce^{-2x} \overset{!}{=} 7e^{-2x}$

It follows that $C = \frac{7}{2}$, so that $y_p = \frac{7}{2}x^2 e^{-2x}$. Hence the general solution is

$$y(x) = \left(C_1 + C_2 x + \frac{7}{2}x^2\right)e^{-2x}.$$

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