Spotlight on the exponential function

Example 65. Solve $y' = ky$ where k is a constant.

Solution. (experience) At this point, we can probably see that $y(x) = e^{kx}$ is a solution.
In fact, the general solution is $y(x) = Ce^{kx}$.
That there cannot be any further solutions follows from the existence and uniqueness theorem (see next example).

Solution. (separation of variables) Alternatively, we can solve the DE using separation of variables.

Express the DE as $\frac{\mathrm{d}y}{\mathrm{d}x} = ky$, then write it as $\frac{1}{y}\mathrm{d}y = k\,\mathrm{d}x$ (note that we just lost the solution $y = 0$).

Integrating gives $\ln|y| = kx + D$, hence $|y| = e^{kx+D}$.

Since the RHS is never zero, $y = \pm e^{kx+D} = Ce^{kx}$ (with $C = \pm e^D$). Finally, note that $C = 0$ corresponds to the singular solution $y = 0$ that we lost. In summary, the general solution is Ce^{kx}.

Example 66. Consider the IVP $y' = ky$, $y(a) = b$. Discuss existence and uniqueness of solutions.

Solution. The IVP is $y' = f(x,y)$ with $f(x,y) = ky$. We compute that $\frac{\partial}{\partial y} f(x,y) = k$.

We observe that both $f(x,y)$ and $\frac{\partial}{\partial y} f(x,y)$ are continuous for all (x,y).

Hence, for any initial conditions, the IVP locally has a unique solution by the existence and uniqueness theorem.

Comment. As a consequence, there can be no other solutions to the DE $y' = ky$ than the ones of the form $y(x) = Ce^{kx}$. Why?! [Assume that $y(x)$ satisfies $y' = ky$ and let (a,b) any value on the graph of y. Then $y(x)$ solves the IVP $y' = ky$, $y(a) = b$; but so does Ce^{kx} with $C = b/e^{ka}$. The uniqueness implies that $y(x) = Ce^{kx}$.]

In particular, we have the following characterization of the exponential function:

e^x is the unique solution to the IVP $y' = y$, $y(0) = 1$.

Comment. Note that, for instance, $\frac{\mathrm{d}}{\mathrm{d}x} 2^x = \ln(2)\, 2^x$. (This follows from $2^x = e^{\ln(2^x)} = e^{x\ln(2)}$.)
Since $\ln = \log_e$, this means that we cannot avoid the natural base $e \approx 2.718$ even if we try to use another base.

Euler's method applied to e^x

Example 67. Consider the IVP $y' = y$, $y(0) = 1$. Approximate the solution $y(x)$ for $x \in [0, 1]$ using Euler's method with 4 steps. In particular, what is the approximation for $y(1)$?

Comment. Of course, the real solution is $y(x) = e^x$. In particular, $y(1) = e \approx 2.71828$.

Solution. The step size is $h = \frac{1-0}{4} = \frac{1}{4}$. We apply Euler's method with $f(x,y) = y$:

$$\begin{array}{ll}
x_0 = 0 & y_0 = 1\\
x_1 = \frac{1}{4} & y_1 = y_0 + h\,f(x_0, y_0) = 1 + \frac{1}{4}\cdot 1 = \frac{5}{4} = 1.25\\
x_2 = \frac{1}{2} & y_2 = y_1 + h\,f(x_1, y_1) = \frac{5}{4} + \frac{1}{4}\cdot\frac{5}{4} = \frac{5^2}{4^2} = 1.5625\\
x_3 = \frac{3}{4} & y_3 = y_2 + h\,f(x_2, y_2) = \frac{5^2}{4^2} + \frac{1}{4}\cdot\frac{5^2}{4^2} = \frac{5^3}{4^3} \approx 1.9531\\
x_4 = 1 & y_4 = y_3 + h\,f(x_3, y_3) = \frac{5^3}{4^3} + \frac{1}{4}\cdot\frac{5^3}{4^3} = \frac{5^4}{4^4} \approx 2.4414
\end{array}$$

In particular, the approximation for $y(1)$ is $y_4 \approx 2.4414$.

Comment. Can you see that, if instead we start with $h = \frac{1}{n}$, then we similarly get $x_i = \frac{(n+1)^i}{n^i}$ for $i = 0, 1, \ldots, n$?
In particular, $y(1) \approx y_n = \frac{(n+1)^n}{n^n} = \left(1 + \frac{1}{n}\right)^n \to e$ as $n \to \infty$. Do you recall how to derive this final limit?

Example 68. (cont'd) Consider the IVP $y' = y$, $y(0) = 1$. Approximate the solution $y(x)$ for $x \in [0, 1]$ using Euler's method with n steps for several values of n. In each case, what is the approximation for $y(1)$?

Solution. Since the real solution is $y(x) = e^x$ so that, in particular, the exact solution is $y(1) = e \approx 2.71828$. We proceed as we did in Example 67 in the case $n = 4$ and apply Euler's method with $f(x, y) = y$:

$$\begin{aligned} x_{n+1} &= x_n + h \\ y_{n+1} &= y_n + h \underbrace{f(x_n, y_n)}_{= y_n} = (1 + h) y_n \end{aligned}$$

We observe that it follows from $y_{n+1} = (1 + h) y_n$ that $y_n = (1 + h)^n y_0$. Since $y_0 = 1$ and $h = \frac{1 - 0}{n} = \frac{1}{n}$, we conclude that

$$x_n = 1, \quad y_n = \left(1 + \frac{1}{n}\right)^n.$$

[For instance, for $n = 4$, we get $x_4 = 1$, $y_4 = \left(\frac{5}{4}\right)^4 \approx 2.4414$ as in Example 67.]

In particular, our approximation for $y(1)$ is $\left(1 + \frac{1}{n}\right)^n$.

Here are a few values spelled out:

$$\begin{aligned} n = 1: &\quad \left(1 + \frac{1}{n}\right)^n = 2 \\ n = 4: &\quad \left(1 + \frac{1}{n}\right)^n = 2.4414\ldots \\ n = 12: &\quad \left(1 + \frac{1}{n}\right)^n = 2.6130\ldots \\ n = 100: &\quad \left(1 + \frac{1}{n}\right)^n = 2.7048\ldots \\ n = 365: &\quad \left(1 + \frac{1}{n}\right)^n = 2.7145\ldots \\ n = 1000: &\quad \left(1 + \frac{1}{n}\right)^n = 2.7169\ldots \\ n \to \infty: &\quad \left(1 + \frac{1}{n}\right)^n \to e = 2.71828\ldots \end{aligned}$$

We can see that Euler's method converges to the correct value as $n \to \infty$. On the other hand, we can see that it doesn't converge impressively fast. That is why, for serious applications, one usually doesn't use Euler's method directly but rather higher-order methods derived from the same principles (such as Runge–Kutta methods).

Interpretation. Note that we can interpret the above values in terms of compound interest. We start with initial capital of $y(0) = 1$ and we are interested in the capital $y(1)$ after 1 year if we receive interest at an annual rate of 100%:

- If we receive a single interest payment at the end of the year, then $y(1) = 2$ (case $n = 1$ above).
- If we receive quarterly interest payments of $\frac{100\%}{4} = 25\%$ each, then $y(1) = (1.25)^4 = 2.441\ldots$ (case $n = 4$).
- If we receive monthly interest payments of $\frac{100\%}{12} = \frac{1}{12}$ each, then $y(1) = 2.6130\ldots$ (case $n = 12$).
- If we receive daily interest payments of $\frac{100\%}{365} = \frac{1}{365}$ each, then $y(1) = 2.7145\ldots$ (case $n = 365$).

It is natural to wonder what happens if interest payments are made more and more frequently. Well, we already know the answer! If interest is compounded continuously, then we have e in our bank account after one year.

Challenge. Can you evaluate the limit $\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$ using your Calculus I skills?