Numerically "solving" DEs: Euler's method

Recall that the general form of a first-order initial value problem is

$$y' = f(x, y), \quad y(x_0) = y_0.$$

Further recall that, under mild assumptions on $f(x, y)$, such an IVP has a unique solution $y(x)$. We have learned some techniques for (exactly) solving DEs. On the other hand, many DEs that arise in practice cannot be solved by these techniques (or more fancy ones).

Instead, it is common in practice to approximate the solution $y(x)$ to our IVP. Euler's method is the simplest example of how this can done. The key idea is to locally approximate $y(x)$ by tangent lines:

Example 55. Suppose y solves the IVP $y' = f(x, y)$ with $y(x_0) = y_0$. Using the tangent line at (x_0, y_0), find an approximation for $y(x_1)$ where $x_1 = x_0 + h$.

The idea is that we choose the **step size** h to be sufficiently small so that the approximation is good enough.

Solution. The tangent line at (x_0, y_0) has slope $y'(x_0) = f(x_0, y_0)$ which is a number we can compute without solving the DE for $y(x)$. Hence, the equation for the tangent line is $T(x) = y_0 + f(x_0, y_0)(x - x_0)$.

We now use this tangent line as an approximation for the solution of the DE to find

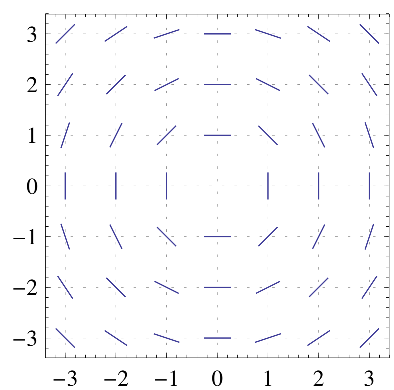
$$y(x_1) \approx T(x_1) = y_0 + f(x_0, y_0)(x_1 - x_0) = y_0 + f(x_0, y_0)h.$$

At this point, we have gone from our initial point (x_0, y_0) to a next (approximate) point (x_1, y_1). We now repeat what we did to get a third point (x_2, y_2) with $x_2 = x_1 + h$. Continuing in this way, we obtain Euler's method:

(Euler's method) To approximate the solution $y(x)$ of the IVP $y' = f(x, y)$, $y(x_0) = y_0$, we start with the point (x_0, y_0) and a step size h. We then compute:

$$\begin{aligned} x_{n+1} &= x_n + h \\ y_{n+1} &= y_n + h f(x_n, y_n) \end{aligned}$$

Example 56. Consider, again, the DE $y' = -x / y$.



We earlier produced the slope field on the right. We also used separation of variables to find that the solutions are circles $y(x) = \pm\sqrt{r^2 - x^2}$.

We know that the unique solution to the IVP with $y(0) = 2$ is $y(x) = \sqrt{4 - x^2}$. On the other hand, approximate the solution using Euler's method with $h = 1$ and 2 steps.

Solution. First, use just the slope field to sketch the 2 approximate points.

On the other hand, applying Euler's method with $f(x, y) = -x / y$:

$$\begin{aligned} &x_0 = 0 && y_0 = 2 \\ &x_1 = 1 && y_1 = y_0 + h f(x_0, y_0) = 2 + 1 \cdot \left(-\frac{0}{2}\right) = 2 \\ &x_2 = 2 && y_2 = y_1 + h f(x_1, y_1) = 2 + 1 \cdot \left(-\frac{1}{2}\right) = \frac{3}{2} \end{aligned}$$

Comment. These are not good approximations! (To get better approximations, the step size must be chosen much smaller.) For comparison, the true values are $y(1) = \sqrt{3} \approx 1.73$ and $y(2) = 0$. Also note that we would get "bogus" values if we take another step to approximate $y(3)$ (whereas the true solution only exists until $x = 2$).

Example 57. Consider the IVP $\frac{dy}{dx} = (2x - 3y)^2 + \frac{2}{3}$, $y(1) = \frac{1}{3}$.

(a) Approximate the solution $y(x)$ for $x \in [1, 2]$ using Euler's method with 2 steps.

(b) Approximate the solution $y(x)$ for $x \in [1, 2]$ using Euler's method with 3 steps.

(c) Solve this IVP exactly. Compare the values at $x = 2$.

Solution.

(a) The step size is $h = \frac{2-1}{2} = \frac{1}{2}$. We apply Euler's method with $f(x, y) = (2x - 3y)^2 + \frac{2}{3}$:

$$\begin{array}{ll} x_0 = 1 & y_0 = \frac{1}{3} \\ x_1 = \frac{3}{2} & y_1 = y_0 + h f(x_0, y_0) = \frac{1}{3} + \frac{1}{2} \cdot \left[\left(2 \cdot 1 - 3 \cdot \frac{1}{3}\right)^2 + \frac{2}{3}\right] = \frac{7}{6} \\ x_2 = 2 & y_2 = y_1 + h f(x_1, y_1) = \frac{7}{6} + \frac{1}{2} \cdot \frac{11}{12} = \frac{13}{8} \end{array}$$

In particular, the approximation for $y(2)$ is $y_2 = \frac{13}{8} = 1.625$.

(b) The step size is $h = \frac{2-1}{3} = \frac{1}{3}$. We again apply Euler's method with $f(x, y) = (2x - 3y)^2 + \frac{2}{3}$:

$$\begin{array}{ll} x_0 = 1 & y_0 = \frac{1}{3} \\ x_1 = \frac{4}{3} & y_1 = y_0 + h f(x_0, y_0) = \frac{1}{3} + \frac{1}{3} \cdot \left[\left(2 \cdot 1 - 3 \cdot \frac{1}{3}\right)^2 + \frac{2}{3}\right] = \frac{8}{9} \\ x_2 = \frac{5}{3} & y_2 = y_1 + h f(x_1, y_1) = \frac{8}{9} + \frac{1}{3} \cdot \frac{2}{3} = \frac{10}{9} \\ x_3 = 2 & y_3 = y_2 + h f(x_2, y_2) = \frac{10}{9} + \frac{1}{3} \cdot \frac{2}{3} = \frac{4}{3} \end{array}$$

In particular, the approximation for $y(2)$ is $y_3 = \frac{4}{3} \approx 1.333$.

(c) We solved this IVP in Example 38 using the substitution $u = 2x - 3y$ followed by separation of variables. We found that the unique solution of the IVP is $y(x) = \frac{2}{3}x - \frac{1}{3(3x-2)}$.

In particular, the exact value at $x = 2$ is $y(2) = \frac{5}{4} = 1.25$.

We observe that our approximations for $y(2) = 1.25$ improved from 1.625 to 1.333 as we increased the number of steps (equivalently, we decreased the step size h from $\frac{1}{2}$ to $\frac{1}{3}$).

For comparison. With 10 steps (so that $h = \frac{1}{10}$), the approximation improves to $y(2) \approx 1.259$.

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