Applications of DEs & Modeling

The exponential model of population growth

If $P(t)$ is the size of a population (eg. of bacteria) at time t, then the rate of change $\frac{\mathrm{d}P}{\mathrm{d}t}$ might, from biological considerations, be (nearly) proportional to $P(t)$.

Why? This might be more clear if we use some (random) numbers. Say, we have a population of $P=100$ and $P'=3$, meaning that the population changes by 3 individuals per unit of time. By how much do we expect a population of $P=500$ to change? (Think about it for a moment!)

[Without further information, we would probably expect the population of $P=500$ to change by $5\cdot 3=15$ individuals per unit of time, so that $P'=15$ in that case. This is what it means for P' to be proportional to P. In formulas, it means that P'/P is constant or, equivalently, that $P'=kP$ for a proportionality constant k.]

Comment. "Population" might sound more specific than it is. It could also refer to rather different populations such as amounts of money (finance) or amounts of radioactive material (physics).

For instance, thinking about an amount $P(t)$ of money in a bank account at time t, we would also expect $\frac{\mathrm{d}P}{\mathrm{d}t}$ (the money per time that we gain from receiving interest) to be proportional to $P(t)$.

The corresponding **mathematical model** is described by the DE $\frac{\mathrm{d}P}{\mathrm{d}t}=kP$ where k is the constant of proportionality.

Example 44. Determine all solutions to the DE $\frac{\mathrm{d}P}{\mathrm{d}t}=kP$.

Solution. We easily guess and then verify that $P(t)=Ce^{kt}$ is a solution. (Alternatively, we can find this solution via separation of variables or because this is a linear DE. Do it both ways!)

Moreover, it follows from the existence and uniqueness theorem that there cannot be further solutions. (Alternatively, we can conclude this from our solving process (separation of variables or our approach to linear DEs only lose solutions when we divide by zero and we can consider those cases separately)).

Mathematics therefore tells us that the (only) solutions to this DE are given by $P(t)=Ce^{kt}$ where C is some constant.

Hence, populations satisfying the assumption from biology necessarily exhibit exponential growth.

The **exponential model** with growth rate k is

$$\frac{\mathrm{d}P}{\mathrm{d}t}=kP.$$

The general solution is $P(t)=Ce^{kt}$ where $C=P(0)$.

Example 45. Let $P(t)$ describe the size of a population at time t. Suppose $P(0)=100$ and $P(1)=300$. Under the exponential model of population growth, find $P(t)$.

Solution. $P(t)$ solves the DE $\frac{\mathrm{d}P}{\mathrm{d}t}=kP$ and therefore is of the form $P(t)=Ce^{kt}$.

We now use the two data points to determine both C and k.

$Ce^{k\cdot 0}=C=100$ and $Ce^{k}=100e^{k}=300$. Hence $k=\ln(3)$ and $P(t)=100e^{\ln(3)t}=100\cdot 3^{t}$.

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The logistic model of population growth

If the population is constrained by resources, then $\frac{dP}{dt} = kP$ is not a good model. A model to take that into account is $\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$. This is the **logistic equation**.

M is called the carrying capacity:

- Note that if $P \ll M$ then $1 - \frac{P}{M} \approx 1$ and we are back to the simpler exponential model. This means that the population P will grow (nearly) exponentially if P is much less than the carrying capacity M.
- On the other hand, if $P > M$ then $1 - \frac{P}{M} < 0$ so that (assuming $k > 0$) $\frac{dP}{dt} < 0$, which means that the population P is shrinking if it exceeds the carrying capacity M.

Comment. If $P(t)$ is the size of a population, then P'/P can be interpreted as its *per capita growth rate*.
Note that in the exponential model we have that $P'/P = k$ is constant.
On the other hand, in the logistic model we have that $P'/P = k(1 - P/M)$ is a linear function.

The **logistic model** with growth rate k and carrying capacity M is

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right).$$

The general solution is $P(t) = \frac{M}{1 + Ce^{-kt}}$ where $C = \frac{M}{P(0)} - 1$.

Important. We will solve the logistic equation in detail in Example 48 to find the stated formula for $P(t)$. At this point, can you already see what technique we will be able to use? (We actually have two options!) Note that, even if we couldn't solve the DE, we can always verify that the stated $P(t)$ solves the DE by plugging in.

Example 46. Let $P(t)$ describe the size of a population at time t. Under the logistic model of population growth, what is $\lim_{t \to \infty} P(t)$?

Solution.

- If $k > 0$, then $e^{-kt} \to 0$ and it follows from $P(t) = \frac{M}{1 + Ce^{-kt}}$ that $\lim_{t \to \infty} P(t) = M$.
  In other words, the population will approach the carrying capacity in the long run.
- If $k = 0$, then we simply have $P(t) = \frac{M}{1 + C}$. In other words, the population remains constant.
  This is a corner case because the DE becomes $\frac{dP}{dt} = 0$.
- If $k < 0$, then $e^{-kt} \to \infty$ and it follows that $\lim_{t \to \infty} P(t) = 0$.
  In other words, the population will approach extinction in the long run.

Comment. There is also the trivial corner case arising from $P(0) = 0$ (then our C would be infinite), in which case $P(t) = 0$. We will always assume that we are not talking about a zero (or negative) population.

Example 47. (homework) A rising population is modeled by the equation $\frac{dP}{dt} = 400P - 2P^2$.

(a) When the population size stabilizes in the long term, how large will it be?

(b) Under which condition would the population size shrink?

(c) What is the population size when it is growing the fastest?

(d) If $P(0) = 10$, what is $P(t)$?

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Solution.

(a) Once the population reaches a stable level in the long term, we have $\frac{\mathrm{d}P}{\mathrm{d}t} = 0$ (no change in population size). Hence, $0 = 400P - 2P^2 = 2P(200 - P)$ which implies that $P = 0$ or $P = 200$. Since the population is rising, it will approach 200 in the long term.

Alternatively. Our DE matches the logistic equation $\frac{\mathrm{d}P}{\mathrm{d}t} = kP\left(1 - \frac{P}{M}\right)$ with $k = 400$ and $M = 200$.

(b) The population size would shrink if $\frac{\mathrm{d}P}{\mathrm{d}t} < 0$.

The DE tells us that is the case if and only if $400P - 2P^2 < 0$ or, equivalently, if $P > \frac{400}{2} = 200$.

Comment. In the logistic model, the population shrinks if it exceeds the carrying capacity.

(c) This is asking when $\frac{\mathrm{d}P}{\mathrm{d}t}$ (the population growth) is maximal.

The DE is telling us that this growth is $f(P) = 400P - 2P^2$. This a parabola that opens to the bottom. From Calculus, we know that it has a global maximum when $f'(P) = 0$.

$f'(P) = 400 - 4P = 0$ leads to $P = 100$.

Thus, the population is growing the fastest when its size is 100.

Comment. In the logistic model, the population is growing fastest when it is half the carrying capacity.

(d) We know that the general solution of the logistic equation is $P(t) = \frac{M}{1 + Ce^{-kt}} = \frac{200}{1 + Ce^{-400t}}$.

Using $P(0) = 10$, we find that $C = \frac{200}{10} - 1 = 19$.

Thus $P(t) = \frac{200}{1 + 19e^{-400t}}$.

Example 48. Solve the logistic equation $\frac{\mathrm{d}P}{\mathrm{d}t} = kP\left(1 - \frac{P}{M}\right)$.

Solution. This is a separable DE: $\frac{1}{P(1 - \frac{P}{M})}\mathrm{d}P = k\mathrm{d}t$.

To integrate the left-hand side, we use partial fractions: $\frac{1}{P(1 - \frac{P}{M})} = \frac{1}{P} + \frac{1/M}{1 - \frac{P}{M}} = \frac{1}{P} - \frac{1}{P - M}$.

After integrating, we obtain $\ln|P| - \ln|P - M| = kt + A$.

Equivalently, $\ln\left|\frac{P}{P - M}\right| = kt + A$ so that $\frac{P}{P - M} = \pm e^{kt + A} = Be^{kt}$ where $B = \pm e^A$.

Solving for P, we conclude that the general solution is

$$P(t) = \frac{BMe^{kt}}{Be^{kt} - 1} = \frac{M}{1 + Ce^{-kt}},$$

where we replaced the free parameter B with $C = -1/B$.

Initial population. Note that the initial population is $P(0) = \frac{M}{1 + C}$. Equivalently, $C = \frac{M}{P(0)} - 1$ which expresses the free parameter C in terms of the initial population.

Comment. Note that $B = \pm e^A$ can be any real number except 0. However, we can easily check that $B = 0$ also gives us a solution to the DE (namely, the trivial solution $P = 0$). This solution was "lost" when we divided by P to separate variables.

Exercise. Note that the logistic equation is also a Bernoulli equation. As an alternative to separation of variables, we can therefore solve it by transforming it to a linear DE via substitution.

Review of partial fractions. Recall that partial fractions tells us that fractions like $\frac{p(x)}{(x - r_1)(x - r_2)\cdots}$ (with the numerator of smaller degree than the denominator; and with the r_j distinct) can be written as a sum of terms of the form $\frac{A_j}{x - r_j}$ for suitable constants A_j.

In our case, this tells us that $\frac{1}{P(1 - P/M)} = \frac{A}{P} + \frac{B}{1 - P/M}$ for certain constants A and B.

Multiply both sides by P and set $P = 0$ to find $A = 1$.

Multiply both sides by $1 - P/M$ and set $P = M$ to find $B = 1/M$. This is what we used above.

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